Section A

Answer **all** questions from this section. Each question carries 8 marks.

1. Find the focus, directrix and axis of the parabola

$$y = x^2 - 2x + 3$$

and sketch the corresponding curve.

[**8**]

2. Prove by induction that every set of $n \geq 0$ elements has exactly 2^n subsets.

[**8**]

3. Using the method of differences to find the sum of the first n terms of the series

$$S_n = 1 + 3x + 5x^2 + 7x^3 + \cdots + (2n-1)(x)^{n-1}.$$

[**8**]

4. Solve the difference equations

(i) $u_{n+1} = 2u_n + n \quad n = 0, 1, 2, \ldots, \quad u_0 = 0$ [**4**]

(ii) $u_{n+2} - 7u_{n+1} + 6u_n = 1, \quad n = 0, 1, 2, \ldots.$ [**4**]

5. A is the matrix given by

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

(i) Find the determinant of A. [**3**]

(ii) Find the inverse of A. [**5**]

6. Find all four complex solutions to

$$z^4 = -4.$$

Identify which solutions form conjugate pairs. [**8**]

Turn over . . .

Section B

Answer **two** questions from this section. Each question carries 26 marks.

7. (a) Find all solutions of the equation

$$2\cos^2 2\theta - \sin 2\theta = 1$$

with $0 \leq \theta \leq 2\pi$, expressing your answer in terms of π.

[10]

(b) Express $\sin 3\theta$ in terms of $\sin\theta$.

[5]

(c) Write down the definition of $\cos^{-1} x$, giving the domain and range of the function.

[3]

(d) Express $\sin(2\cos^{-1} x)$ in terms of x only.

[8]

8. (a) Give the definitions of symmetric, reflexive and transitive relations.

[4]

A relation $\sim$ is defined on the set $A = \{a, b, c\}$.

Say whether the following relations are symmetric, reflexive and/or transitive, giving reasons:

(i) $\sim = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$, **[3]**

(ii) $\sim = \{(a, b), (a, c), (b, a), (b, c), (c, a), (c, b)\}$, **[3]**

(iii) $\sim = \{(a, a), (a, b), (b, a), (b, b)\}$. **[3]**

(b) (i) A relation $\sim$ is defined on the complex numbers $\mathbf{C}$ by $z_1 \sim z_2$ if $|z_1 - 1| = |z_2 - 1|$. Show that $\sim$ is an equivalence relation on $\mathbf{C}$. Describe geometrically the equivalence classes of $\sim$. **[7]**

(ii) We define a relation $<$ on the complex numbers $\mathbf{C}$ by $z_1 < z_2$ if $|z_1 - 1| < |z_2 - 1|$. Is this (1) an equivalence relation, (2) a partial ordering, (3) a total ordering. In each case give your reasons.

[6]

Turn over ...

9. (a) Write the following system of equations in matrix form:

$$\begin{aligned} x + y + z &= 1, \\ 2x + 3y + 5z &= 5, \\ 4x + 3y + a^2 z &= a, \end{aligned}$$

where a is a constant. By considering the rank of the resulting matrix and augmented matrix, say how many solutions there are for this system of equations for the three cases $a = -1$, 0 and 1. Do *not* calculate the solution to the equations when possible. **[13]**

(b) Using row manipulation methods, find the inverse of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 3 & 0 \end{pmatrix}.$$

[13]

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