Problems with Inverting Matrices: The Cost of Beer

One Friday lunchtime, a group of students went to a local pub for a (liquid) lunch. Unfortunately, only one of them had any money. But this student kindly offered to pay for them all on the condition that they refunded the money at a later time. During the course of this "lunch" four rounds were purchased, each consisting of combinations of bitter, lager, cider and stout. Due to an oversight the generous student made a record of who had drunk how much of each beer and what the total cost of each round was, but failed to make a record of how much a pint of each of the beers was! Fortunately the student diligently paid attention in lectures and realised that it was possible to calculate the cost of the beers from the information available ...

The available data are summarised in the table below

Round no.	Pints of beer bought				Cost of Round
	Bitter	Lager	Cider	Stout	
1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	$\frac{1}{2}$	£8.99
2	$2\frac{1}{2}$	$\frac{1}{2}$	2	$2\frac{1}{2}$	£10.46
3	2	$2\frac{1}{2}$	3	1	£11.78
4	1	2	0	$2\frac{1}{2}$	£7.99

The student set the problem up in matrix form, denoting the costs of pints of bitter, lager, cider and stout as w, x, y and z respectively, giving

$$\begin{pmatrix} 1\frac{1}{2} & 2 & 2\frac{1}{2} & \frac{1}{2} \\ 2\frac{1}{2} & \frac{1}{2} & 2 & 2\frac{1}{2} \\ 2 & 2\frac{1}{2} & 3 & 1 \\ 1 & 2 & 0 & 2\frac{1}{2} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8.99 \\ 10.46 \\ 11.78 \\ 7.99 \end{pmatrix}$$

The student was aware of the problems that could occur if the matrix was not invertible, and checked that the determinant was indeed non-zero. Thus in full confidence the student proceeded to calculate the cost of each beer. The results were

A pint of bitter costs	£–3.86
A pint of lager costs	£0.60
A pint of cider costs	£4.58
A pint of stout costs	£4.26

The student was puzzled! On checking the calculations no errors were found! The charges made by the publican were also correct.

If nobody made any errors why does it look like the students were being paid to drink bitter? What were the correct costs of the drinks?

To be continued ....