Problems with Inverting Matrices: The Cost of Beer

The Solution

The true costs of the drinks were

A pint of bitter cost	£1.23
A pint of lager cost	£1.44
A pint of cider cost	£1.39
A pint of stout cost	£1.55

On ringing up the prices in the till the publican had followed the standard practice of rounding up the cost of the half pints to the nearest penny. This resulted in the first round costing an extra $1\frac{1}{2}$p, the second round 1p and the last round a halfpenny. The third round did not experience any upward rounding. Although these changes were small they resulted in big changes in the estimated costs.

The matrix

$$\begin{pmatrix} 1\frac{1}{2} & 2 & 2\frac{1}{2} & \frac{1}{2} \\ 2\frac{1}{2} & \frac{1}{2} & 2 & 2\frac{1}{2} \\ 2 & 2\frac{1}{2} & 3 & 1 \\ 1 & 2 & 0 & 2\frac{1}{2} \end{pmatrix}$$

does indeed have a non-zero determinant, its determinant is 1/16. Loosely speaking, the matrix is almost singular! This results in the elements of the inverse matrix having very large elements. The inverse of this matrix is

$$\begin{pmatrix} -316 & -22 & 278 & -26 \\ -52 & -4 & 46 & -4 \\ 198 & 14 & -174 & 16 \\ 168 & 12 & -148 & 14 \end{pmatrix}.$$

You have to multiply this inverse matrix by the vector made up of the costs of the rounds in order to find the costs of the beers. Any small variations in the costs of the rounds such as would be caused by rounding the cost of the half pints up could lead to an error which is comparable to the variation in the price charged times the size of the largest entries in the inverse matrix. With entries of the magnitude that are encountered in this inverse this error will have a large effect of the calculated cost of the individual beers.

The answer given at the top of the page is the only possible one. If you assume that only one of the different drinks has an odd price for a pint then the estimated costs of the individual drinks are

Odd priced drink	Estimated cost of drinks			
	Bitter	Lager	Cider	Stout
bitter	£–2.17	£0.88	£3.52	£3.36
lager	£–5.14	£0.39	£5.38	£4.94
cider	£–2.28	£0.86	£3.59	£3.42
stout	£–2.04	£0.90	£3.44	£3.29

It is clear from the above table that having a pint of lager which costs an odd number of pennies is not a good idea as it makes matters worse! We will assume from now on that its price will always be an even number of pennies. If we now try out the possibility that only two of the others have an odd price we find the estimates are

Odd priced drinks	Estimated cost of drinks			
	Bitter	Lager	Cider	Stout
bitter and cider	£–0.59	£1.14	£2.53	£2.52
bitter and stout	£–0.35	£1.18	£2.38	£2.39
cider and stout	£–0.46	£1.16	£2.45	£2.45

Although these estimates are getting better, the price of bitter is still negative! The only way that you can get it positive is if the costs of bitter, cider and stout are all odd numbers of pennies, while lager is an even number of pennies. Then you obtain the answer given at the beginning of this sheet.

The moral of this is to be careful when calculating results. In general, problems such as this do not arise frequently, but they can occur. If you have large variations in your calculated answers that come from small variations in your input data then you may have a problem that is "almost singular". It could be that your problem is in reality singular, but that errors in the coefficients in the matrix has made it seem nonsingular. In such cases you will have to reformulate your problem to avoid the singularity. If the problem is just badly behaved as in the above problem, the all you can do is measure your data *very* carefully.