Mathematics: Term 2 Questions 10
Further Integration

1. Evaluate the following integrals:

   (i) $\displaystyle\int_0^\infty e^{-3x}\sin x\,dx$

   (ii) $\displaystyle\int_0^\infty e^{-x}\cos 2x\,dx$

   (iii) $\displaystyle\int_0^\pi \cos^6 x\,dx$

   (iv) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$

   (v) $\displaystyle\int_0^1 x^5 e^{-2x}\,dx$

2. If

$$I_n = \int_0^\pi x^n \cos x\,dx,$$

   show that

$$I_n = -n\pi^{n-1} - n(n-1)I_{n-2}.$$

   Hence evaluate

$$\int_0^\pi x^4 \cos x\,dx.$$

3. If

$$I_n = \int_0^{\pi/2} x\sin^n x\,dx,$$

   show that

$$I_n = \frac{1}{n^2} + \frac{(n-1)}{n} I_{n-2}.$$

   Hence evaluate

$$\int_0^\pi x\sin^5 x\,dx.$$

4. Find the lengths of the curves given over the intervals indicated:

   (i) $y^2 = x^3$, between $x = y = 0$ and $x = y = 1$.

   (ii) $y = \ln\sec x$, between $x = 0$ and $x = \pi/6$.

5. Find the area of the surface obtained when the curve $y = 2x^{1/2}$ between $x = y = 0$ and $x = y = 4$ is rotated around the x-axis.

Solutions

1. (i) 1/10
   (ii) 1/5
   (iii) $5\pi/16$
   (iv) 0 Note: any integral of the form $\int_{-a}^{a} f(x)\,dx$, where $f(x)$ is an odd function will be 0.
   (v) $\frac{15}{8} - \frac{109e^{-2}}{8}$

2. $I_n = \int_0^{\pi} x^n \cos x\,dx = [x^n \sin x]_0^{\pi} - \int_0^{\pi} nx^{n-1} \sin x\,dx$
   $= 0 + \left[nx^{n-1}\cos x\right]_0^{\pi} - \int_0^{\pi} n(n-1)x^{n-2}\cos x\,dx = -n\pi^{n-1} - n(n-1)I_{n-2}.$
   $I_4 = -4\pi^3 - 12I_2 = -4\pi^3 - 12(-2\pi - 2I_0) = -4\pi^3 + 24\pi$ as $I_0 = 0$.

3. $I_n = \int_0^{\pi/2} x \sin^n x\,dx = \int_0^{\pi/2} x(1-\cos^2 x)\sin^{n-2} x\,dx = I_{n-2} - \int_0^{\pi/2} (x\cos x)(\cos x \sin^{n-2} x)\,dx$
   $= I_{n-2} - \left[(x\cos x)\frac{\sin^{n-1} x}{n-1}\right]_0^{\pi/2} + \int_0^{\pi/2} (\cos x - x\sin x)\frac{\sin^{n-1} x}{n-1}\,dx$
   $= I_{n-2} - 0 + \int_0^{\pi/2} \cos x \frac{\sin^{n-1} x}{n-1}\,dx - \frac{1}{n-1}I_n$
   $= I_{n-2} - 0 + \left[\frac{\sin^n x}{n(n-1)}\right]_0^{\pi/2} - \frac{1}{n-1}I_n = I_{n-2}\frac{1}{n(n-1)} - \frac{1}{n-1}I_n$. Hence $I_n = \frac{1}{n^2} + \frac{(n-1)}{n}I_{n-2}$.
   $I_5 = \frac{1}{25} + \frac{4}{5}I_3 = \frac{1}{25} + \frac{4}{5}\left(\frac{1}{9} + \frac{2}{3}I_1\right)$. But $I_1 = 1$, so $I_5 = 149/225$

4. (i) $\frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right) = \frac{13\sqrt{13} - 8}{27}$.
   (ii) $\frac{\ln 3}{2}$.

5. $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} = \left(1 + \left(\frac{1}{x^{1/2}}\right)^2\right)^{1/2} = \left(\frac{x+1}{x}\right)^{1/2}$, so

$$\text{Area} = 2\pi \int_0^4 2x^{1/2}\left(\frac{x+1}{x}\right)^{1/2}\,dx = 2\pi \int_0^4 2\,(x+1)^{1/2}\,dx$$

$$= 4\pi\left[\frac{2}{3}(x+1)^{3/2}\right]_0^4 = \frac{8\pi}{3}\left(5\sqrt{5} - 1\right).$$