Mathematics: Term 2 Questions 4
Complex Numbers

In the following z (with or without a subscript) is always complex and can be expressed as $z = x+iy$ with x and y real. Questions 6–8 were not covered explicitly in the lectures, but should be possible with a bit of thought.

1. Perform the following operations, expressing your answer in the form$a + ib$ with a and b real:

   (a) $(2+3i)+(1+4i)$,   (b) $(6-2i)+(-3+4i)$,

   (c) $(3+2i)-(-2+i)$,   (d) $(2+3i)\times(1-2i)$,

   (e) $(2+4i)\times(-1+2i)$,   (f) $\dfrac{i(2+3i)}{1-2i}$.

2. Find

   (a) $\mathrm{Re}(2-3i)\times(1+2i)$,   (b) $\mathrm{Re}\left(\dfrac{1}{1+i}\right)$,

   (c) $\mathrm{Im}\,\dfrac{(2-3i)^2}{2+3i}$,   (d) $\mathrm{Im}\,\dfrac{z}{\bar{z}}$,

3. Represent in polar form

   (a) -4,   (b) $1+i$,

   (c) $\dfrac{1+i}{1-i}$,   (d) $\dfrac{9+13i}{1+3i}$,

   (e) $\dfrac{3\sqrt{2}+2i}{-\sqrt{2}-2i/3}$.   (f) $\dfrac{(1+i)(2+i)}{3-i}$,

4. Verify $r(\cos\theta + i\sin\theta)\times r'(\cos\theta' + i\sin\theta') = rr'(\cos(\theta+\theta') + i\sin(\theta+\theta'))$.

   Use this result to prove by induction the formula of De Moivre:

   $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

   Hence find an expression for $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

5. Find all five solutions to $z^5 = 1$, and show them graphically.

6. What part of the complex plane is represented by the inequality

   $$|z+1| > |z-1|$$

   Give both an algebraic proof and a geometrical argument.

7. Show that if $z_1 z_2 = 0$ then at least one of z_1 and z_2 is zero.

8. Demonstrate that $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$.

Solutions

1. (a) $3+7$ (b) $3+2i$
(c) $5+i$ (d) $8-i$
(e) -10 (f) $-\frac{7}{5}-\frac{4i}{5}$

2. (a) 8 (b) $\frac{1}{2}$
(c) $-\frac{9}{13}$ (d) $\frac{2xy}{x^2+y^2}$

3. (a) $4(\cos\pi + i\sin\pi)$ (b) $\sqrt{2}(\cos\pi/4 + i\sin\pi/4)$
(c) $i = 1(\cos\pi/2 + i\sin\pi/2)$ (d) $5(\cos\alpha + i\sin\alpha),\ \alpha = -\tan^{-1}\frac{7}{24}$
(e) $-3 = 3(\cos\pi + i\sin\pi)$ (f) $i = 1(\cos\pi/2 + i\sin\pi/2)$

4. $$r(\cos\theta + i\sin\theta)\times r'(\cos\theta' + i\sin\theta')$$
$$= rr'(cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\sin\theta\cos\theta' + \sin\theta'\cos\theta) = rr'(\cos(\theta+\theta') + i\sin(\theta+\theta')).$$

Statement in question is obviouslly true for $n=1$. If true for n then from above result (with $r=r'=1$)
$$(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)\times(\cos\theta + i\sin\theta)^n$$
$$= (\cos\theta + i\sin\theta)\times(\cos n\theta + i\sin n\theta) = \cos(n+1)\theta + i\sin(n+1)\theta.$$
Hence it is also true for $n+1$, and by induction true for all n.
$$\sin 5\theta = \mathrm{Im}(\cos 5\theta + i\sin 5\theta) = \mathrm{Im}\left((\cos\theta + i\sin\theta)^5\right)$$
$$= \mathrm{Im}\left(\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\right)$$
$$= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

5. Can use previous result with $z = \cos\theta + i\sin\theta$ to show $z^5 = 1 \Rightarrow 5\theta = 2n\pi$. Hence $\theta = 0, \pm\frac{2}{5}\pi, \pm\frac{4}{5}\pi, \ldots$. These points give the corners of a pentagon whose corners lie on the unit circle, with one corner at 1.

6. $|z+1| > |z-1| \Rightarrow |z+1|^2 > |z-1|^2 \Rightarrow (z+1)(\overline{z}+1) > (z-1)(\overline{z}-1)$
$\Rightarrow z+\overline{z} > -z-\overline{z} \Rightarrow 2(z+\overline{z}) > 0 \Rightarrow \mathrm{Re}\, z > 0$

Geometrically this inequality is satisfied by all points that are closer in the complex plane to 1 than to -1, hence it gives all points in the right-half plane.

7. $z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) = 0$. Taking real and imaginary parts gives
$$(1)\quad x_1x_2 - y_1y_2 = 0 \qquad \text{and} \qquad (2)\quad x_1y_2 + x_2y_1 = 0.$$
Assume, without loss of generality, that $z_1 \neq 0$. Then adding $x_1\times(1)$ to $y_1\times(2)$ gives $x_2(x_1^2 + y_1^2) = 0$. Since $z_1 \neq 0$ this implies $x_2 = 0$. Similary subtracting $y_1\times(1)$ from $x_1\times(2)$ gives $y_2(x_1^2+y_1^2) = 0$ and so $y_2 = 0$. Hence if $z_1 \neq 0$ then $z_2 = 0$ as required.

8. $$\overline{z_1/z_2} = \overline{\frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2} + i(\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2})} = \frac{x_1x_2 + y_1y_2}{x_2^2+y_2^2} - i(\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2})$$
$$= \frac{(x_1 - iy_1)(x_2 + iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{x_1 - iy_1}{x_2 - iy_2} = \overline{z_1}/\overline{z_2}.$$