Mathematics: Term 2 Questions 5
Series

1. Use the method of differences to find the sum the first n terms of the following series

   (a) $S_n = 1 + 2x + 3x^2 + 4x^3 + \cdots$

   (b) $S_n = 4 + 6x + 8x^2 + 10x^3 + \cdots$

   (c) $S_n = 0 + 3x + \cdots + (i^2 - 1)x^{i-1} + \cdots$

   (d) $S_n = 1 + 2^2x + 3^2x^2 +^2 x^3 + \cdots$

2. In parts (a) and (d) of question 1 you could try and find the sums $1 + 2 + 3 + \cdots + n$ and $1 + 2^2 + 3^2 + \cdots + n^2$ by setting $x = 1$, but it doesn't work. Why? However, once you have found S_n to can recover the correct expressions by finding

$$\lim_{x \to 1} S_n.$$

   Verify that this works.

3. Which of the following series converge, giving your reasons

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{3n+2}$

   (b) $\displaystyle\sum_{n=0}^{\infty} n \left(\frac{1+\sqrt{2}}{3} \right)^n$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{n!}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solutions

1. (a) $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$

   (b) $\dfrac{4-2x-(4+2n)x^n+2(n+1)x^{n+1}}{(1-x)^2}$

   (c) $\dfrac{3x-x^2-n(n+2)x^n+(2n^2+2n-3)x^{n+1}-(n^2-1)x^{n+2}}{(1-x^3)}$

   (d) $\dfrac{1+x-(1+n)^2x^2+2n(n+1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}$

2. Setting $x=1$ means that when you multiply the series by x and subtract it from the original series you get $(1-x)S_n=0$. So S_n is no longer in your equation.

3. (a) $(n+2)/(3n+2)\to 1/3$ as $n\to\infty$, since the individual terms do not tend to zero the series cannot converge.

   (b) By the ratio test, $|a_{n+1}/a_n| = |\frac{(n+1)(1+\sqrt{2})}{3n}| \to \frac{1+\sqrt{2}}{<}1$ so the series converges.

   (c) By the ratio test $a_{n+1}/a_n = \frac{(n+1)^{2(n+1)}}{(n+1)!}\times\frac{n!}{n^{2n}} = \frac{(n+1)^{2n+1}}{n^{2n}} = (\frac{n+1}{n})^{2n}(n+1)$. But $\frac{n+1}{n}>1$ and so $a_{n+1}/a_n > n+1 > 1$ for all n. Hence the series diverges.

   (d) Use the integral test. As

$$\int \frac{1}{x\ln x}\,dx = \ln(\ln x) + C \to \infty \quad \text{as} \quad x\to\infty$$

   so the series must diverge.