Computational Mathematics/Information Technology

Dr Oliver Kerr

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Introduction

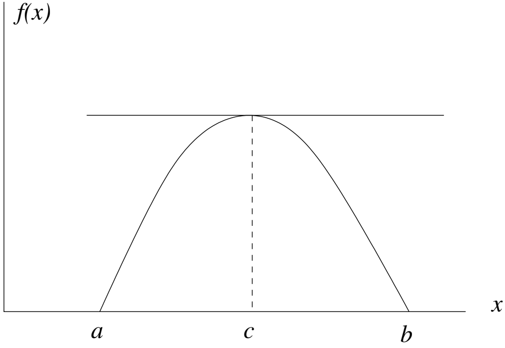
In the presentation on numerical integration the result

$$f(a+x) = f(a) + xf'(a) + x^2 f''(a+\theta(x)x)/2$$

for some $0 < \theta(x) < 1$ was used. Here we will derive this result using Rolle's theorem.

Rolle's Theorem

If a real function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the openinterval (a, b) and $f(a) = f(b) = 0$ then for some $c \in (a, b)$ we have $f'(c) = 0$.



To prove this we make use of the **Extreme Value Theorem** that says a function $f(x)$ which is continuous on the closed interval $[a, b]$ is bounded, and attains its bounds. I.e. there is a least upper bound M so that $f(x) \leq M$ in the interval, and there is a c such that $f(c) = M$. Similarly for a lower bound.

This theorem is not proved here — it needs things like the completeness of the real line.

Since $f(a) = f(b) = 0$, if $M > 0$ then for $\delta > 0$

$$\frac{f(c+\delta) - f(c)}{\delta} < 0$$

and so

$$\lim_{\delta \to 0+} \frac{f(c+\delta) - f(c)}{\delta} \le 0$$

Similarly, $\delta < 0$

$$\frac{f(c+\delta) - f(c)}{\delta} > 0$$

and so

$$\lim_{\delta \to 0-} \frac{f(c+\delta) - f(c)}{\delta} \ge 0$$

As $f(x)$ is differentialbe, these two limits must exist and be the same, i.e. $f'(c)$. As we have $f'(c) \ge 0$ and $f'(c) \le 0$ it is clear $f'(c) = 0$.

Note: If $M = 0$ (no positive maximum) then consider $g(x) = -f(x)$. If neither $f(x)$ nor $g(x)$ have positive maximums then $f(x)$ is constant and zero, and so has zero derivative everywhere.

Back to

$$f(a+x) = f(a) + xf'(a) + x^2 f''(a+\theta(x)x)/2$$

Consider

$$\phi(t) = f(a+t) - f(a) - tf'(a) - Bt^2$$

where we choose B so that $\phi(h) = 0$.

Since $\phi(0) = \phi(h) = 0$, by Rolle's theorem there is an h_1 with $0 < h_1 < h$ such that $\phi'(h_1) = 0$.

Now

$$\phi'(t) = f'(a+t) - f'(a) - 2Bt$$

But $\phi'(0) = 0$ and $\phi'(h_1) = 0$, so applying Rolle's theorem to $\phi'(t)$ it tells us there is h_2, with $0 < h_2 < h_1 < h$ where $\phi''(h_2) = 0$.

Lastly

$$\phi''(t) = f''(a+t) - 2B$$

Since $\phi''(h_2) = 0$ we have

$$B = f''(a+h_2)/2$$

for some h_2 with $0 < h_2 < h$.

From $\phi(h) = 0$ we get

$$0 = f(a+h) - (f(a) + hf'(a) + h^2 f''(a+h_2)/2)$$

which gives us our result.