Mathematical Methods: Complex Variables 2

Answer Sheet

1. (a) From the definition

$$\frac{d}{dz}\{f(z)+g(z)\} = \lim_{\Delta z\to 0}\frac{\{f(z+\Delta z)+g(z+\Delta z)\}-\{f(z)+g(z)\}}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} + \lim_{\Delta z\to 0}\frac{g(z+\Delta z)-g(z)}{\Delta z} = \frac{d}{dz}f(z)+\frac{d}{dz}g(z)$$

(b) Let $p = g(z)$ and $w = f(p)$, then $w = f(g(z))$. Let a change δz in z correspond to a change δp in p, and a corresponding change δw in w, i.e. $p + \delta p = g(z + \delta z)$, and $w + \delta w = f(p + \delta p)$. Then since f and g are differentiable

$$\delta p = (g'(z)+\epsilon_1)\delta x, \qquad \text{where} \quad \epsilon_1 \to 0 \quad \text{as} \quad \delta z \to 0,$$

$$\delta w = (f'(p)+\epsilon_2)\delta p, \qquad \text{where} \quad \epsilon_2 \to 0 \quad \text{as} \quad \delta p \to 0.$$

Hence

$$\delta w = (f'(p)+\epsilon_2)(g'(z)+\epsilon_1)\delta z.$$

As $\delta z \to 0$, so $\epsilon_1 \to 0$. This implies that $\delta p \to 0$, and hence $\epsilon_2 \to 0$. Hence

$$\lim_{\delta z\to 0}\frac{\delta w}{\delta z} = \lim_{\delta z\to 0}(f'(p)+\epsilon_2)(g'(z)+\epsilon_1) = f'(p)g(z) = f'(g(z))g'(z).$$

As this limit exists and is independent of the argument of δz then $f(g(z))$ is analytic.

2. (a) $u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$ hence $u_x = 3x^2 - 3y^2 = v_y, \quad u_y = -6xy = -v_x$. $f(z)$ is analytic.

(b) $u = 0, \quad v = x^2 + y^2$, hence $u_x = 0 \neq v_y = 2y, \quad u_y = 0 \neq -v_x = -2x$. $f(z)$ is not analytic anywhere, but is differentiable at $z = 0$.

(c) $u = \arctan(y/x), \quad v = 0$, hence $u_x = -y/(x^2+y^2) \neq v_y = 0, \quad u_y = x/(x^2+y^2) \neq -v_y = 0$. $f(z)$ is not analytic, nor differentiable anywhere.

(d) Here

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}, \qquad \text{and} \qquad v = -\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

Hence

$$u_x = \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = -v_y, \qquad \text{and} \qquad u_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} - v_x.$$

And so $f(z)$ is analytic. $[f(z) = \cot z]$

3. $|f(z)|^2 = u(x,y)^2 + v(x,y)^2 = C$, where C is a constant. Taking partial derivatives with respect to x and y gives

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0, \qquad \text{and} \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Use the Cauchy-Riemann equations to eliminate the partial derivatives of v:

$$uu_x - vu_y = 0, \qquad uu_y + vu_x = 0.$$

Then add u times the first equation to v times the second equation to eliminate u_y, giving

$$(u^2 + v^2)u_x = 0.$$

Hence $u_x = v_y = 0$. Similarly $u_y = -v_x = 0$, and so $f(z)$ is a constant.

4. If $f(z) = u(x,y) + iv(x,y)$ then $\overline{f}(z) = u(x,-y) - iv(x,-y) = U(x,y) + iV(x,y)$. Then $U_x(x,y) = u_x(x,-y)$, $U_y(x,y) = -u_y(x,-y)$, $V_x(x,y) = -v(x,-y)$, and $V_y(x,y) = v_y(x,-y)$. Applying the Cauchy-Riemann equations on u and v gives $U_x = V_y$, and $U_y = -V_x$, and so U and V satisfy the Cauchy-Riemann equations for all z and so $\overline{f}(z)$ is analytic.

5. Consider the point $(x,y) = r(\cos\theta, \sin\theta)$, then

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = r\left((1+i)\sin^3\theta - (1-i)\cos^3\theta\right), \quad \text{for all} \quad r \neq 0.$$

Clearly as $z \to 0$ then $r \to 0$ and $f(z) \to 0$. Hence $f(z)$ is continuous at the origin. Also

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2}, \qquad \text{and} \qquad v(x,y) = \frac{x^3 + y^3}{x^2+y^2},$$

and so

$$\frac{\partial}{\partial x}u(0,0) = \lim_{x\to 0}\frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1$$

Similarly

$$\frac{\partial}{\partial y}u(0,0) = -1, \qquad \frac{\partial}{\partial x}v(0,0) = 1, \qquad \frac{\partial}{\partial y}v(0,0) = 1$$

and so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at} \quad x = y = 0.$$

But

$$\frac{f(z) - f(0)}{z} = \frac{r^3\cos^3\theta(1+i) - r^3\sin^3\theta(1-i)}{r^3\cos\theta + i\sin\theta}$$
$$= (1+i)(\cos^4\theta + \sin^4\theta) + (1-i)\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$

which is clearly dependent on θ, and so $f'(0)$ does not exist. So what went wrong? The problem comes from not being able to differentiate either $u(x,y)$ and $v(x,y)$ in both variables together, i.e. we cannot find constants A and B such that

$$u(x,y) = u(0,0) + Ax + By + o(r).$$

6. (a) $\nabla^2(y^2-x^2) = \frac{\partial^2}{\partial x^2}(x^2-y^2) + \frac{\partial^2}{\partial y^2}(y^2-x^2) = 2-2 = 0$. If $u = y^2-x^2$ then $u_x = -2x = v_y \Rightarrow v = -2xy + g(z)$, and $u_y = 2y = -v_x = 2y - g'(x) \Rightarrow g'(x) = 0$, and so $g(x) = C$, for some constant C, hence $v = -2xy + C$.

   (a) $f(z) = y^2 - x^2 + i(-2xy + C) = -(x+iy)^2 + iC$. Quite quick.

   (b) $f(z) = \frac{1}{(2i)^2}(z-\overline{z})^2 - \frac{1}{2^2}(z+\overline{z})^2 - \frac{2i}{(2)(2i)}(z+\overline{z})(z-\overline{z}) + iC = -\frac{1}{4}z^2 + \frac{1}{2}z\overline{z} - \frac{1}{4}\overline{z}^2 - \frac{1}{4}z^2 - \frac{1}{2}z\overline{z} - \frac{1}{4}\overline{z}^2 - \frac{1}{2}z^2 + \frac{1}{2}\overline{z}^2 + iC = -z^2 + iC$. Quite tedious.

   (c) $f(z) = 0 - z^2 + i(-2z0 + C) = -z^2 + iC$. Quick.

   (d) $f(z) = 2\left[(-iz/2)^2 - (z/2)^2\right] + C = 2[-z^2/2 - z^2] + C = -z^2 + C$. Quick if you can remember the formula (since you don't have to find $v(x,y)$).

   (b) $\nabla^2(e^x \cos y) = \frac{\partial^2}{\partial x^2}(e^x \cos y) + \frac{\partial^2}{\partial y^2}(e^x \cos y) = e^x \cos y + (-e^x \cos y) = 0$. If $u = e^x \cos y$ then $u_x = e^x \cos y = v_y \Rightarrow v = e^x \sin y + g(x)$, and $u_y = -e^x \sin y = -v_x = -e^x \sin y - g'(x) \Rightarrow g'(x) = 0$, hence $v = e^x \sin y + C$.

   (a) $f(z) = e^x \cos y + ie^x \sin y + iC = e^x(e^{iy} + e^{-iy})/2 + ie^x(e^{iy} - e^{-iy})/(2i) + iC = e^{x+iy}/2 + e^{x-iy}/2 + e^{x+iy}/2 - e^{x-iy}/2 + iC = e^z + iC$.

   (b) Very long winded, I'm not prepared to type it out.

   (c) $f(z) = e^z \cos 0 + ie^z \sin 0 + iC = e^z + iC$. Quick.

   (d) $f(z) = 2e^{z/2}\cos(-iz/2) + C = e^{z/2} 2\cosh(z/2) + C = e^{z/2}\left(e^{z/2} + e^{-z/2}\right) = e^z + 1 + C = e^z + C'$.

7. $\nabla^2(x^3 - 3xy^2 - 2x) = \frac{\partial^2}{\partial x^2}(x^3 - 3xy^2 - 2x) + \frac{\partial^2}{\partial y^2}(x^3 - 3xy^2 - 2x) = 6x - 6x = 0$. If $u = x^3 - 3xy^2 - 2x$ then $u_x = 3x^2 - 3y^2 - 2 = v_y \Rightarrow v = 3x^2y - y^3 - 2y + g(x)$, and $u_y = -6xy = -v_x = -6xy - g'(x) \Rightarrow g'(x) = 0$, hence $v = 3x^2y - y^3 - 2y + C$. Using $f(z) = u(z,0) + iv(z,0)$ gives $f(z) = z^3 - 3z \times 0 - 2z + i(3z^2 \times 0 - 0^3 + C = z^3 - 2z + iC$.

   For last part see notes.

8. [Note error in question: it should read $\sin x$ and not $\sin z$ in the definition of $u(x,y)$]

   $\nabla^2(x + \sin x \cosh y) = \frac{\partial^2}{\partial x^2}(x + \sin x \cosh y) + \frac{\partial^2}{\partial y^2}(x + \sin x \cosh y) = -\sin x \cosh y + \sin x \cosh y = 0$. If $u = x + \sin x \cosh y$ then $u_x = 1 + \cos x \cosh y = v_y \Rightarrow v = y + \cos x \sinh y + g(x)$, and $u_y = \sin x \sinh y = -v_x = \sin x \sinh y - g'(x) \Rightarrow g'(x) = 0$, hence $v = y + \cos x \sinh y + C$. Using $f(z) = u(z,0) + iv(z,0)$ gives $f(z) = z + \sin z \cosh 0 + i(0 + \cos z \sinh 0 + C) = z + \sin z + iC$.

   For last part see notes.

9. To see if it is analytic we check to see if its derivative exists. Note:

$$\frac{1}{z+\Delta z} - \frac{1}{z} = \frac{-\Delta z}{(z+\Delta z)z}$$

and so

$$\lim_{\Delta z\to 0}\frac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z}=\lim_{\Delta z\to 0}\frac{-1}{(z+\Delta z)z}=-\frac{1}{z^2}.$$

This limit exists for all $z\neq 0$, hence $1/z$ is analytic for all $z\neq 0$.

From the chain rule (Question 1), if $h(z)=1/z$, and $q(z)$ is a polynomial (and hence analytic) then $h(q(z))=1/q(z)$ is analytic, except when $q(z)=0$. The product of two analytic functions is also analytic, so $p(z)h(q(z))=p(z)/q(z)$ is analytic as required.

10. (a) $\cos z=\cos x\cosh y-i\sin x\sinh y$, hence if $\cos z$ is to be real then $\sin x\sinh y=0$. This is the case when either $\sin x=0$, in which case $z=n\pi+iy$ for any integer n and arbitrary y, or when $\sinh y=0$, in which case $y=0$ and so z is any real number.

    (b) $\sin z=\sin x\cosh y+i\cos x\sinh y$, hence if $\sin z$ is real then $\cos x\sinh y=0$. If $\cos x=0$ then $z=(n+1/2)\pi+iy$ for any integer n and arbitrary real y, and if $\sinh y=0$ the z is real number.

11. Using the definitions $\cos z=(e^{iz}+e^{-iz})/2$ and $\sin z=(e^{iz}-e^{-iz})/(2i)$

$$\cos^2 z+\sin^2 z=\frac{(e^{iz}+e^{-iz})^2}{2^2}+\frac{(e^{iz}-e^{-iz})^2}{(2i)^2}=\frac{e^{2iz}}{4}+\frac{1}{2}+\frac{e^{-2iz}}{4}-\frac{e^{2iz}}{4}+\frac{1}{2}-\frac{e^{-2iz}}{4}=1$$

$$\cos(-z)=\left(e^{i(-z)}+e^{-i(-z)}\right)/2=\left(e^{-iz}+e^{iz}\right)/2=\cos z$$

$$\sin(-z)=\left(e^{i(-z)}-e^{-i(-z)}\right)/(2i)=\left(e^{-iz}-e^{iz}\right)/(2i)=-\left(e^{iz}-e^{-iz}\right)/(2i)=-\sin z$$

$$\tan(-z)=\frac{\sin(-z)}{\cos(-z)}=\frac{-\sin z}{+\cos z}=-\tan z$$

$$\cos(z_1\pm z_2)=\left(e^{iz_1\pm iz_2}+e^{-iz_1\mp iz_2}\right)/2=\left(e^{iz_1}e^{\pm iz_2}+e^{-iz_1}e^{\mp iz_2}\right)/2$$
$$=\left((\cos z_1+i\sin z_1)(\cos z_2\pm\sin z_2)+(\cos z_1-i\sin z_1)(\cos z_2\mp i\sin z_2)\right)/2$$
$$=\cos z_1\cos z_2\mp\sin z_1\sin z_2$$

$$\sin(z_1\pm z_2)=(e^{iz_1\pm iz_2}-e^{-iz_1\mp iz_2})/(2i)=(e^{iz_1}e^{\pm iz_2}-e^{-iz_1}e^{\mp iz_2})/(2i)$$
$$=\left((\cos z_1+i\sin z_1)(\cos z_2\pm i\sin z_2)-(\cos z_1-i\sin z_1)(\cos z_2\mp i\sin z_2)\right)/(2i)$$
$$=\sin z_1\cos z_2\pm\cos z_1\sin z_2$$

12. From the definition of a general power of a complex number

$$i^i=e^{i\ln i}=e^{i\ln(e^{i\pi/2})}=e^{i(i\pi/2+2n\pi i)}=e^{-\pi/2}e^{-2n\pi},$$

where n is any integer.