Mathematical Methods: Complex Variables 3

Answer Sheet

1. (a) $\nabla^2(xy+x+y) = \frac{\partial^2}{\partial x^2}(xy+x+y) + \frac{\partial^2}{\partial y^2}(xy+x+y) = 0+0=0$. Harmonic.

   (b) If $u(x,y) = \frac{y}{(x+1)^2+y^2}$ then $u_x = \frac{-2(x+1)y}{((x+1)^2+y^2)^2}$, $u_{xx} = \frac{6(x+1)^2y-2y^3}{((x+1)^2+y^2)^3}$, and $u_y = \frac{(x+1)^2-y^2}{((x+1)^2+y^2)^2}$, $u_{xx} = -\frac{6(x+1)^2y-2y^3}{((x+1)^2+y^2)^3}$. Hence $\nabla^2 u(x,y) = u_{xx}+u_{yy} = 0$. Harmonic.

   (c,d) If $u(x,y) = (y\cos x \mp x\sin x)e^{-y}$ then $u_x = (-y\sin x \mp (x\cos x + \sin x))e^{-y}$, $u_{xx} = (-y\cos x \mp (-x\sin x + 2\cos x))e^{-y}$, and $u_y = (\cos x - y\cos x \mp (-x\sin x))e^{-y}$, $u_{yy} = (y\cos x - 2\cos x \mp (x\sin x))e^{-y}$. Hence $\nabla^2 u(x,y) = (\mp 2\cos x - 2\cos x)e^{-y}$ and so (c) $\nabla^2 u(x,y) = -4\cos x\, e^{-y}$. Not Harmonic. (d) $\nabla^2 u(x,y) = 0$. Harmonic.

   For (a) $u_x = y+1 = v_y \Rightarrow v = \frac{1}{2}y^2 + y + g(x)$

   $u_y = x+1 = -v_x = -g'(x) \Rightarrow g(x) = -\frac{1}{2}x^2 - x + C$.

   Hence $v(x,y) = \frac{1}{2}(y^2-x^2) + y - x + C$ and $f(z) = u(z,0) + iv(z,0) = z - \frac{1}{2}iz^2 - iz + iC = (1-i)z - \frac{i}{2}z^2 + iC$.

   For (b) $u_x = \frac{-2(x+1)y}{((x+1)^2+y^2)^2} = v_y \Rightarrow v = \frac{(x+1)}{(x+1)^2+y^2} + g(x)$

   $u_y = \frac{(x+1)^2-y^2}{((x+1)^2+y^2)^2} = -v_x = \frac{(x+1)^2-y^2}{((x+1)^2+y^2)^2} + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = C$.

   Hence $v(x,y) = \frac{x+1}{(x+1)^2+y^2} + C$ and $f(z) = u(z,0) + iv(z,0) = 0 + i\frac{z+1}{(z+1)^2+0} + iC = \frac{i}{z+1} + iC$.

   Note (c) is not harmonic so it is NOT the real part of some analytic function.

   For (d) $u_x = -y\sin x\, e^{-y} + \sin x\, e^{-y} + x\cos x\, e^{-y} = v_y \Rightarrow v = y\sin x\, e^{-y} - x\cos x\, e^{-y} + g(x)$.
   $u_y = (-y\cos x + \cos x - x\sin x)e^{-y} = -v_x = -(y\cos x - \cos x + x\sin x)e^{-y} - g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = C$.

   Hence $v(x,y) = (y\sin x - x\cos x)e^{-y} + C$ and $f(z) = u(z,0) + iv(z,0) = z\sin z - iz\cos z$. This can be further simplified: $f(z) = z\sin z - iz\cos z = \frac{z}{2i}\left(e^{iz} - e^{-iz}\right) - \frac{iz}{2}\left(e^{iz} + e^{-iz}\right) = -ize^{iz}$.

2. (a) i. Splitting the contour into two parts

   $$\int_0^2 \overline{z}\, dz = \int_0^2 x\, dx = \left[\frac{x^2}{2}\right]_0^2 = 2$$

   $$\int_2^{2+2i} \overline{z}\, dz = \int_0^2 (2-iy)\, idy = \left[2iy + \frac{y^2}{2}\right]_0^2 = 2+4i.$$

   Giving the total integral of $4+4i$.

   ii. With $z = x+iy = 2t + 2it^2$, $dz = (2+4it)dt$, and so

   $$\int_C \overline{z}\, dz = \int_0^1 \overline{z}\frac{dz}{dt}dt = \int_0^1 (2t - 2it^2)(2+4it)dt$$

   $$= \int_0^1 4t + 4it^2 + 8t^3\, dt = \left[2t^2 + \frac{4it^3}{3} + 2t^4\right]_0^1 = 4 + \frac{4i}{3}.$$

(b) i. Again splitting the contour into two parts

$$\int_0^2 z^2 + 2iz\,dz = \int_0^2 x^2 + 2ix\,dx = \left[\frac{x^3}{3} + ix^2\right]_0^2 = \frac{8}{3} + 4i$$

$$\int_2^{2+2i} z^2 + 2iz\,dz = \int_0^2 (2+iy)^2 + 2i(2+iy)\,idy = i\int_0^2 4(1+i) + (4i-2)y - y^2\,dy$$

$$= i\left[4(1+i)y + (2i-1)y^2 - \frac{y^3}{3}\right]_0^2 = i\left(16i + \frac{4}{3}\right) = -16 + \frac{4}{3}i$$

Giving the total integral of $-13\frac{1}{3} + 5\frac{1}{3}i$.

ii. Again with $z = x + iy = 2t + 2it^2$, $dz = (2+4it)dt$:

$$\int_C z^2 + 2iz\,dz = \int_0^1 (z^2 + 2iz)\frac{dz}{dt}\,dt = \int_0^1 \left[(2t+2it^2)^2 + 2i(2t+2it^2)\right](2+4it)\,dt$$

$$= \int_0^1 8it - 16t^2 + 16it^3 - 40t^4 - 16it^5\,dt$$

$$= \left[4it^2 - \frac{16}{3}t^3 + 4it^4 - 8t^5 - \frac{8i}{3}t^6\right]_0^1 = -13\tfrac{1}{3} + 5\tfrac{1}{3}i.$$

Note for integral (b) the function is analytic, and the answers *should* be the same. Finding the indefinite integral

$$\int z^2 + 2iz\,dz = \frac{z^3}{3} + iz^2 + C,$$

we can check these integral: $\left[z^3/3 + iz^2 + C\right]_0^2 = \frac{(2+2i)^3}{3} + i(2+2i)^2 = -13\frac{1}{3} + 5\frac{1}{3}i$ as required.

3. The formula given holds if the function $f(z)$ is analytic inside and on a closed simple contour C that encloses the point z_0. This contour is traversed in an anticlockwise direction.

Applying Cauchy's Integral Formula

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z)^{n+1}}dz'\right| \le \frac{n!}{2\pi}\left|\oint_C \frac{f(z')}{(z'-z)^{n+1}}dz'\right|,$$

where C is the circle of radius r centred on z_0. But on this circle $|z - z_0| = r$ and $|f(z)| \le M$ and so

$$\left|\oint_C \frac{f(z')}{(z'-z)^{n+1}}dz'\right| \le \frac{M}{r^{n+1}} \times 2\pi r = \frac{2\pi M}{r^n},$$

giving

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \times \frac{2\pi M}{r^n} = \frac{n!M}{r^n}.$$

If $f(z) = e^z$ then $|f(z)| = e^x$ where x is the real part of z. Hence on the circle $|z| = 1$ the maximum modulus of $f(z)$ will correspond to the point with largest real part, i.e. $z = 1$, and so $M = e$. This gives an upper bound for $|f^{(n)}(0)|$ of

$$|f^{(n)}(0)| \le n!e.$$

We know that $f^{(n)}(0) = 1$ for all n, however the upper bound derived here grows as $n!$, so although this method produces a reasonable bound for the first couple of of derivatives, the bound rapidly becomes very weak for larger values of n and may not be of much practical use.

If a function is entire then it is analytic for all z, and we can apply the above result for any circle of arbitrary radius and centre. If a function is globally bounded in modulus by M', say, then using the above result applied to a circle of radius r centred on z we find that the modulus of the derivative of $f(z)$ bounded by

$$|f'(z)| \leq \frac{M}{r}.$$

But the radius can be chosen to be as large as we want, and so this implies that $|f'(z)| = 0$. This is true for all points, and so $f'(z) = 0 \quad \forall z$. Hence $f(z)$ is constant.

The *real* function $f(x) = \cos x$ is indeed bounded everywhere and is not constant, but the above result is a result for *complex analytic* functions. The complex function that corresponds to the real function $\cos x$, i.e. $\cos z$, has a modulus that grows roughly as $\frac{1}{2}e^{|y|}$ as y, the imaginary part of z, gets large in magnitude. Hence the complex function $\cos z$ is clearly not globally bounded in the same way.

4. Cauchy's Integral Formula states that if a function is analytic inside and on a simple closed contour C, and a point z lies within C then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz',$$

where the contour is traversed in the anticlockwise direction. For proof see notes. The formula for higher derivatives is

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z' - z)^{n+1}} dz'.$$

(a) Note $z^2 + 1 = (z + i)(z - i)$, so we will have to deform the contour into two separate contours, C_1 and C_2, surrounding the singularities at $z = i$ and $z = -i$ respectively. Around $z = i$ we see that

$$\frac{\cosh 2z}{z^2 + 1} = \frac{\cosh 2z}{z + i} \times \frac{1}{z - i}.$$

If the contour around $z = i$ doesn't contain $z = -i$ then the first fraction is analytic and we can apply Cauchy's Integral Theorem. If we let

$$g(z) = \frac{\cosh 2z}{z + i}$$

then

$$g(i) = \frac{\cosh 2i}{i + i} = \frac{1}{2\pi i} \int_{C_1} \frac{g(z')}{z' - i} dz',$$

and so

$$\int_{C_1} \frac{\cosh 2z}{z^2+1} dz = (2\pi i) \times \frac{\cosh 2i}{2i}.$$

Similarly the contribution from around the singularity at $z = -i$ is

$$\int_{C_2} \frac{\cosh 2z}{z^2+1} dz = (2\pi i) \times \frac{\cosh -2i}{-2i} = -(2\pi i) \times \frac{\cosh 2i}{2i},$$

and so the contribution from each singularity cancels the other out and the total integral is 0. This result could be anticipated from the symmetry of the argument of the integral: observe $\cosh 2z/(z^2+1) = \cosh(-2z)/((-z)^2+1)$ and so contributions from opposite side of the circle will cancel.

(b) Again we will split the contour of integration into three parts around each of the singularities, C_1 about $z = 0$, C_2 about $z = i$ and C_3 about $z = -i$. Then if we set $g(z) = \frac{\sin 2z}{z^2+1}$, then $g(z)$ is analytic in the neighbourhood of $z = 0$ and we can apply Cauchy's Integral Formula for the first derivative

$$g'(0) = \frac{1}{2\pi i} \int_{C_1} \frac{g(z')}{z'^2} dz'.$$

Giving

$$\int_{C_1} \frac{\sin 2z}{z^2(z^2+1)} dz = 2\pi i \times g'(0) = 2\pi i \times \left. \frac{2(z^2+1)\cos 2z - 2z \sin 2z}{(z^2+1)^2} \right|_{z=0} = 2\pi i \times 2 = 4\pi i.$$

In a similar fashion to part (a)

$$\int_{C_2} \frac{\sin 2z}{z^2(z^2+1)} dz = \int_{C_2} \frac{\sin 2z}{z^2(z+i)} \times \frac{1}{z-i} dz = 2\pi i \times \frac{\sin 2i}{i^2(i+i)} = -\pi \sin 2i$$

and

$$\int_{C_3} \frac{\sin 2z}{z^2(z^2+1)} dz = \int_{C_3} \frac{\sin 2z}{z^2(z-i)} \times \frac{1}{z+i} dz = 2\pi i \times \frac{\sin(-2i)}{(-i)^2(-i-i)} = -\pi \sin 2i.$$

Giving the total integral

$$\int_C \frac{\sin 2z}{z^2(z^2+1)} dz = 4\pi i - \pi \sin 2i - \pi \sin 2i = 2\pi(2i - \sin 2i) \quad [= 2\pi i(2 - \sinh 2)].$$

5. From Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z'-z} dz',$$

where C is a simple contour going around the point z anticlockwise. Deform the contour to a circle of radius r centred on a. Then $z' = a + re^{i\theta}$, and $dz' = ire^{i\theta} d\theta$. Giving

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z'-a} dz' = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{a+re^{i\theta}-a} ire^{i\theta} d\theta = \int_0^{2\pi} f\left(a+re^{i\theta}\right) d\theta,$$

as required.

6. Assume that there is a point, z', with maximum modulus in the interior of the region under consideration. Let r be the radius of a circle centred on the point such that all points on the circle lie inside or on C. Then from Gauss' mean Value Theorem

$$f(z') = \frac{1}{2\pi}\int_0^{2\pi} f(z' + re^{i\theta})\, d\theta.$$

And so

$$|f(z')| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z' + re^{i\theta})\, d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z' + re^{i\theta})\right|\, d\theta.$$

If M is the maximum modulus of $f(z)$ for any point on this circle then

$$|f(z')| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z' + re^{i\theta})\right|\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} M\, d\theta = M.$$

But $|f(z')|$ has a global maximum at z', and so $|f(z')| \geq M$, hence $|f(z')| = M$. This equality is only possible if

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(z' + re^{i\theta})\right|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} M\, d\theta.$$

Because $f(z)$ is continuous this can only be the case if $|f(z)| = M$ for all points on the circle. Since r is arbitrary this implies that the modulus of $f(z)$ is constant in any disc centred on z' inside or on the boundary, and will hold for the largest possible circles that just touches the boundary, hence there must be a point on the boundary at which $|f(z)| = M$. And so the maximum modulus is to found on the boundary too. Hence result.

7. A function $f(z)$ is analytic inside and on a boundary C, and is not zero for any point inside or on the boundary (if it was zero on the boundary the minimum modulus theorem is trivially true). Then consider $g(z) = 1/f(z)$, this function is analytic for all points inside and on the boundary, and so from Q6 its maximum modulus is attained on the boundary. However, the maximum modulus of $g(z)$ corresponds to the minimum modulus of $f(z)$, hence result.

8. If a line γ_1 is given by $z = g_1(t)$, then the angle the curve makes with the real axis at any point is given by $\arg\left(\frac{dz}{dt}\right) = \arg g_1'(t)$. Similarly if another curve is given by $z = g_2(s)$ then the angle this makes with the real axis is $\arg\left(\frac{dz}{ds}\right) = \arg g_2'(s)$. If the curves meet when $t = t_0$ and $s = s_0$ then the angle between the curves will be $\arg g_1'(t_0) - \arg g_2'(s_0) = \arg[g_1'(t_0)/g_2'(s_0)]$. The angle the corresponding curves will make in the w-plane are $\arg\left(\frac{d\,f(z)}{dt}\right) = \arg\left[f'(g_1(t))g_1'(t)\right]$, and $\arg\left(\frac{d\,f(z)}{ds}\right) = \arg\left[f'(g_2(s))g_2'(s)\right]$. The angle between these two lines at the point of intersection will be

$$\arg\left[f'(g_1(t_0))g_1'(t_0)\right] - \arg\left[f'(g_2(s_0))g_2'(s_0)\right] = \arg\left[\frac{f'(g_1(t_0))g_1'(t_0)}{f'(g_2(s_0))g_2'(s_0)}\right].$$

But $g_1(t_0) = g_2(s_0)$ and so the terms involving the derivatives of f cancel, giving the same angle as was found between the lines in the z-plane.

9. (a) See notes.

   (b) See notes.

   (c) From a point $z = re^{i\theta}$ consider a point a distance δr in the radial direction, i.e. $z = re^{i\theta} + \delta re^{i\theta}$. Then considering the limit as $\delta r \to 0$

$$f'(z) = \lim_{\delta r \to 0} \frac{f(z + \delta re^{i\theta}) - f(z)}{\delta re^{i\theta}}$$

$$= \lim_{\delta r \to 0} \frac{u(r + \delta r, \theta) + iv(r + \delta r, \theta) - u(r, \theta) - iv(r, \theta)}{\delta re^{i\theta}}$$

$$= \frac{u_r(r, \theta) + iv_r(r, \theta)}{e^{i\theta}}.$$

Similarly, if we consider the derivative as a limit as $z + \delta z$ approaches z along the circular arc $|z + \delta z| = r$, i.e. let $z + \delta z = re^{i(\theta+\delta\theta)}$ then

$$f'(z) = \lim_{\delta\theta \to 0} \frac{f(re^{i(\theta+\delta\theta)}) - f(re^{i\theta})}{r(e^{i(\theta+\delta\theta)} - e^{i\theta})}$$

$$= \frac{u(r, \theta + \delta\theta) + iv(r, \theta + \delta\theta) - u(r, \theta) - iv(r, \theta)}{r(e^{i(\theta+\delta\theta)} - e^{i\theta})}$$

But $e^{i(\theta+\delta\theta)} - e^{i\theta} = ie^{i\theta}\delta\theta + O(\delta\theta^2)$ and so

$$f'(z) = \frac{u_\theta(r, \theta) + iv_\theta(r, \theta)}{ire^{i\theta}}$$

These must be the same, hence

$$\frac{u_r(r, \theta) + iv_r(r, \theta)}{e^{i\theta}} = \frac{u_\theta(r, \theta) + iv_\theta(r, \theta)}{ire^{i\theta}}$$

or

$$u_r(r, \theta) + iv_r(r, \theta) = \frac{1}{ir}(u_\theta(r, \theta) + iv_\theta(r, \theta)).$$

Taking real and imaginary parts gives the required result

$$ru_r = v_\theta \qquad \text{and} \qquad rv_r = -u_\theta.$$

Lastly, if $v(r, \theta) = r^{-2} \cos 2\theta$ then

$$v_r = -\frac{2}{r^3} \cos 2\theta = -\frac{u_\theta}{r} \Rightarrow u = \frac{\sin 2\theta}{r^2} + g(r)$$

$$u_r = -\frac{2 \sin 2\theta}{r^3} + g'(r) = \frac{v_\theta}{r} = -\frac{2 \sin 2\theta}{r^3} \Rightarrow g'(r) = 0 \Rightarrow g(r) = C.$$

Hence

$$u(r, \theta) = \frac{\sin 2\theta}{r^2} + C$$

and

$$f(z) = u(r, \theta) + iv(r, \theta) = \frac{\sin 2\theta + i \cos 2\theta}{r^2} + C = \frac{ie^{-2i\theta}}{r^2} + C = \frac{i}{r^2 e^{2i\theta}} = \frac{i}{z^2}.$$