Mathematical Methods: Complex Variables 4

Answer Sheet

1. Assume $P(z)$ has no roots then $1/P(z)$ will have no singularities. Also, since $n \geq 1$, $1/P(z) \to 0$ as $z \to \infty$, and so $1/P(z)$ will be bounded in magnitude. Hence by Liouville's theorem $1/P(z)$ is a constant. This is clearly not the case, therefor our assumption that $P(z)$ has no roots must be false, and that there exists at least one point, z_1 say, such that $P(z_1) = 0$.

   Since $P(z_1) = 0$ then $P(z) = P(z) - P(z_1) = a_1(z - z_1) + a_2(z^2 - z_1^2) + a_3(z^3 - z_1^3) + \cdots + a_n(z^n - z_1^n)$. We can take out a factor of $(z - z_1)$ from each term to give $P(z) = (z - z_1)[a_1 + a_2(z + z_1) + a_3(z^2 + zz_1 + z_1^2) + \cdots + a_n(z^{n-1} + z^{n-2}z_1 + \cdots + z_1^{n-1})]$. The expression inside the brackets $[\ldots]$ is a polynomial of order $n - 1$, hence we have shown that $P(z) = (z - z_1)Q(z)$, where Q(z) is a polynomial of order $n - 1$. This process can be repeated to show that $Q(z)$ has a root, z_2, and so $P(z) = (z - z_1)(z - z_2)R(z)$ where $R(z)$ is a polynomial of order $n - 2$. We continue repeating this process until we are left with $P(z) = (z - z_1)(z - z_2)(z - z_3)\ldots(z - z_n)C$, where C is a constant. Examination of the coefficient of z^n shows us that $C = a_n$ as required. From this expression it is clear that $P(z)$ has n roots.

2. (a) i. α is a removable singularity, means that $\lim_{z\to\alpha} f(z)$ exists, and if $f(\alpha)$ is redefined too take this value at the point α then $f(z)$ becomes analytic at α.

   ii. α is a pole, means that there exists a positive integer n and a constant $A \neq 0$ such that $\lim_{z\to\alpha} f(z)(z - \alpha)^n = A$.

   iii. α is an essential singularity means that if $f(z)$ is a single-valued function (i.e. α is not a branch point) and that the point α is not analytic, but it is neither a pole nor a removeable singularity, then it is an essential singularity.

   Note that the last two singularities can also be defined in terms of their Laurent series. If the Laurent series has a finite number of non-zero coefficients of the negative powers, then the singularity is a pole. If it has an infinite number of such coefficients then the point is an essential singularity.

   (b) In case (i) if $\lim_{z\to\alpha} \neq 0$ then the singularity of $1/f(z)$ is also removable. If the limit is 0, then the singularity is a pole.

   In case (ii) the singularity will be a removable singularity, it can be replaced with the value 0.

   In case (iii) the singularity of $1/f(z)$ is also an essential singularity.

   (c) The function $f(z) = \frac{e^{1/z}(z-\pi)}{\sin z}$ has singularities at $z = 0$, $\pm\pi$, $\pm2\pi, \ldots$. From consideration of the Laurent series for $e^{1/z} = 1 + z^{-1} + z^{-2}/2! + z^{-3}/3! + \cdots$, it is clear that there are an infinite number of terms in this series about $z = 0$ with negative powers, and so $z = 0$ is an essential singularity. The singularity at $z = \pi$ corresponds to both $(z - \pi) = 0$ and $\sin z = 0$. As $z \to \pi$, $f(z) = \frac{e^{1/z}(z-\pi)}{\sin z} \to -e^{1/\pi}$, and so this is a removable singularity. All the other singularities are simple poles.

3. $a_n = \sqrt{n+1} - \sqrt{n} = \sqrt{n}(\sqrt{1+1/n} - 1)$. But $1 + \epsilon \leq 1 + \epsilon + \epsilon^2/4$ and so $\sqrt{1+\epsilon} \leq 1 + \epsilon/2$, hence $a_n \leq \sqrt{n}(1 + 1/(2n) - 1) = (2\sqrt{n})^{-1} \to 0$ as $n \to 0$. However if $S_N = \sum_0^N a_n = 1 - 0 + \sqrt{2} - 1 + \cdots + \sqrt{N} - \sqrt{N-1} + \sqrt{N+1} - \sqrt{N} = \sqrt{N+1}$ and so $S_N \to \infty$ as $N \to \infty$.

4. (a) $(n+2)/(3n+2i) \to 1/3$ as $n \to \infty$, since the individual terms do not tend to zero the series cannot converge.

   (b) By the ratio test, $|a_{n+1}/a_n| = |\frac{(n+1)(1+i)}{2n}| \to \left|\frac{1+i}{2}\right| = 1/\sqrt{2} < 1$ so the series converges.

   (c) By the ratio test $a_{n+1}/a_n = \frac{(n+1)^{2(n+1)}}{(n+1)!} \times \frac{n!}{n^{2n}} = \frac{(n+1)^{2n+1}}{n^{2n}} = (\frac{n+1}{n})^{2n}(n+1)$. But $\frac{n+1}{n} > 1$ and so $a_{n+1}/a_n > n + 1 > 1$ for all n. Hence the series diverges.

   (d) Both the root test and the ratio test fail to say whether this series diverges or converges. Probably the easiest way to see that this series converges is to consider the real and imaginary parts separately. In both cases you get a sequence of numbers converging to zero with alternating signs, and hence each subseries converges and so the whole series converges.

5. Consider $\lim_{n\to\infty} \sqrt[n]{|a_n(z-z_0)^n|} = \lim_{n\to\infty} |z - z_0| \sqrt[n]{|a_n|} = L|z - z_0|$. By the root test, if $L|z - z_0| > 1$, i.e. if $|z - z - 0| > 1/L$ then the series diverges, similarly if $L|z - z_0| < 1$, i.e. $|z - z_0| < 1/L$, the series converges. This means that the radius of convergence is $1/L$.

   It is probably easiest to see this works by using a "hand-waving" argument. If the sequence has more than one limit point then all the points that we need to worry about in this sequence will have values in some neighbourhoods of these limit points. If we then separate the terms in the series into several power series according to which limit point the terms $\sqrt[n]{|a_n|}$ are near then from the above argument we see that each of the separate series will have different radii of convergence. The series with the largest radius of convergence will be the one with the lowest vale of $\lim_{\to\infty} \sqrt[n]{|a_n|}$, and the smallest radius of convergence will have the largest value. For the power to converge as a whole the individual power series must also converge, and this will be limited by the smallest radius of convergence of the individual power series. This will correspond to the largest of the limit points of the sequence $\{\sqrt[n]{|a_n|}\}$.

   Examination of the coefficients in the given power series shows that for even n the general term is $(2+2^{-n})$ and so $\sqrt[n]{|a_n|} = 1$. For odd n the general term is 2^{-n}, and so $\sqrt[n]{|a_n|} = 1/2$. The larger of these is 1, so the radius of convergence of this power series is $1/1 = 1$.

When applying the above and the root test you may find it useful to know that for large n, $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ (Stirling's formula), and so $\sqrt[n]{n!} \approx n/e$.

6. (a) $\frac{d^n}{dz^n}\log(z+1) = (-1)^{n-1}(n-1)!/(z+1)^n$ and so $\log(z+1) = z - z^2/2 + z^3/3 - z^4/4 + \cdots - (-z)^n/n - \cdots$

   (b) $\frac{d^n}{dz^n}(z+1)^{-1} = (-1)^n n!/(z+1)^{n+1}$ and so $1/(z+1) = 1 - z + z^2 - z^3 + z^4 + \cdots + (-z)^n + \cdots$

   (c) $\frac{d^n}{dz^n}(-1/(z+1)^2) = (-1)^{n-1}(n+1)!/(z+1)^{n+2}$ and so $-1/(z+1)^2 = -1 + 2z - 3z^2 + 4z^3 - 5z^4 \cdots - (n+1)(-z)^n - \cdots$

   It is straight forward to show that if you integrate the series term-by-term (b) you get the series (a), and if you differentiate it term-by-term you get the series (c).

7. (a) We can either use the result from 6(b) or alternatively:

   $\frac{1}{1+z} = \frac{1}{2}(1+\frac{z-1}{2})^{-1} = 1/2 - (z-1)/4 + (z-1)^2/8 - (z-1)^3/16 + \cdots + (-1)^n(z-1)^n/2^{n+1} + \cdots$, $R = 2$.

   (b) It is easiest in this case to use partial fractions:

$$\frac{1}{1+z^2} = 1/2\left[\frac{1}{1-iz} + \frac{1}{1+iz}\right] = \frac{1}{2(1-i)}\frac{1}{1-\frac{i(z-1)}{1-i}} + \frac{1}{2(1+i)}\frac{1}{1+\frac{i(z-1)}{1+i}}$$

$$= \frac{1}{2(1-i)}\left[1 + \frac{i(z-1)}{1-i} + \frac{i^2(z-1)^2}{(1-i)^2} + \frac{i^3(z-1)^3}{(1-i)^3} + \cdots\right]$$

$$+ \frac{1}{2(1+i)}\left[1 - \frac{i(z-1)}{1+i} + \frac{i^2(z-1)^2}{(1+i)^2} - \frac{i^3(z-1)^3}{(1+i)^3} + \cdots\right]$$

$$= 2^{-3/2}e^{i\pi/4}\left[1 + 2^{-1/2}e^{3i\pi/4}(z-1) + 2^{-1}e^{6i\pi/4}(z-1)^2 + 2^{-3/2}e^{9i\pi/4}(z-1)^3 + \cdots\right]$$

$$+ 2^{-3/2}e^{-i\pi/4}\left[1 + 2^{-1/2}e^{-3i\pi/4}(z-1) + 2^{-1}e^{-6i\pi/4}(z-1)^2 + 2^{-3/2}e^{-9i\pi/4}(z-1)^3 + \cdots\right]$$

$$= 2^{-3/2}[e^{i\pi/4} + e^{-i\pi/4}] + 2^{-2}[e^{i\pi} + e^{-i\pi}](z-1) + 2^{-5/2}[e^{7i\pi/4} + e^{-7i\pi/4}](z-1)^2$$

$$+ 2^{-3}[e^{10i\pi/4} + e^{-10i\pi/4}](z-1)^3 + 2^{-7/2}[e^{13i\pi/4} + e^{-13i\pi/4}](z-1)^4 + \cdots$$

$$= \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 - \frac{1}{8}(z-1)^4 + \frac{1}{8}(z-1)^5 - \frac{1}{16}(z-1)^6 + \frac{1}{32}(z-1)^8 + \cdots$$

   (c) This one gets a bit tedious:

$$f(z) = \sin \pi z^2 \Rightarrow f(2) = 0$$

$$f'(z) = 2\pi z \cos \pi z^2 \Rightarrow f'(2) = 4\pi$$

$$f''(z) = 2\pi \cos \pi z^2 - 4\pi^2 z^2 \sin \pi z^2 \Rightarrow f''(2) = 2\pi$$

$$f'''(z) = -8\pi^3 z^3 \cos \pi z^2 - 12\pi^2 z \sin \pi z^2 \Rightarrow f'''(2) = -64\pi^3$$

$$f''''(z) = -48\pi^3 z^2 \cos \pi z^2 + (16\pi^4 z^4 - 12\pi^2)\sin \pi z^2 \Rightarrow f''''(2) = -192\pi^4$$

   and so on. Hence $f(z) = 4\pi(z-2) + \pi(z-2)^2 - \frac{32}{3}\pi(z-2)^3 - 8\pi^3(z-2)^4 + \left(\frac{128}{15}\pi^5 - 2\pi^3\right)(z-2)^5 + \cdots$.

   (d) $\ln 2z = \ln(2(z-2)+4) = \ln 4 + \ln(1 + \frac{(z-2)}{2}) = \ln 4 + \frac{(z-2)}{2} - \frac{(z-2)^2}{4} + \frac{(z-2)^3}{24} - \frac{(z-2)^4}{64} + \cdots$

8. Remembering

$$a_n = \frac{1}{2\pi i}\oint_C \frac{1}{1-z} \times \frac{1}{(z+1)^n}\,dz \tag{1}$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{1}{1-z} \times (z+1)^{n-1}\,dz \tag{2}$$

where C is an appropriate contour surrounding $z_0 = -1$. For the Laurent series near the centre we can use any circle of radius less than 2 centred on 1. In this case the singularity of the $1/(1-z)$ term is not included in the contour, and hence the integrand for (2) is analytic inside C, and so $b_n = 0$ for all n. In (1) the only singularity is that at $z = -1$. This is a pole of order n. Using Cauchy's integral formula this gives

$$a_n = \frac{1}{2\pi i}\oint_C \frac{1}{1-z} \times \frac{1}{(z+1)^n}\,dz = \frac{1}{2\pi i} \times 2\pi i \frac{1}{(n-1)!}\frac{d^{(n-1)}}{dz^{(n-1)}}\frac{1}{1-z}\bigg|_{z=-1}$$

$$= \frac{1}{(n-1)!} \times (n-1)!(1-z)^{-n}\bigg|_{z=-1} = (n-1)! \times \frac{(n-1)!}{2^n} = \frac{1}{2^n}$$

Hence the Laurent/Taylor series for $|z+1| < 2$ is

$$1/(1-z) = 1 + \frac{(z+1)}{2} + \frac{(z+1)^2}{4} + \frac{(z+1)^3}{8} + \cdots$$

For C of radius larger than 2 integral we still get the above contribution for the coefficients a_n, but in addition we get a contribution from the singularity at $z = 1$. From Cauchy's integral formula we find this extra bit

$$\frac{1}{2\pi i}\oint_{C_1} \frac{1}{1-z} \times \frac{1}{(z+1)^n}\,dz = \frac{1}{2\pi i} \times -2\pi i \frac{1}{(z+1)^n}\bigg|_{z=1} = -\frac{1}{2^n}.$$

Thus $a_n = 2^{-n} - 2^{-n} = 0$. (2) now contains the singularity at $z = 1$. From Cauchy's integral formula

$$b_n = \frac{1}{2\pi i}\oint_C \frac{1}{1-z} \times \frac{1}{(z+1)^n}\,dz = \frac{1}{2\pi i} \times -2\pi i (z+1)^{n-1}\bigg|_{z=1} = -2^{n-1}$$

Hence for $|z+1| > 2$ the Laurent series is $1/(1-z) = -1 - \frac{2}{(z+1)} - \frac{4}{(z+1)^2} - \frac{4}{(z+1)^3} - \cdots$

9. (a) $\frac{1}{z-3} = -\frac{1}{3(1-z/3)} = z^{-1}\frac{1}{1-3/z}$. To find the Laurent series for $|z| < 3$ (i.e. the Taylor series) expand the second expresseion, and for $|z| > 3$ the third expression to give:

For $|z| < 3$, $1/(z-3) = -1/3 - z/9 - z^2/27 - z^3/81 - z^4/243 - \cdots$.

For $|z| > 3$, $1/(z-3) = z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + 81z^{-5} + \cdots$.

(b) Use partial fractions $1/[z(z-1)(z-2)] = \frac{1}{2}z^{-1} - (z-1)^{-1} + \frac{1}{2}(z-2)^{-1}$. We then expand the relevant terms according to whether (i) $|z| < 1$, (ii) $1 < |z| < 2$ or (iii) $2 < |z|$.

(i) $= \frac{1}{2}z^{-1} + [1 + z + z^2 + z^3 + z^4 + \cdots] + \left[-\frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \cdots\right] = \frac{1}{2}z^{-1} + \frac{3}{4} + \frac{7}{8}z + \frac{15}{16}z^2 + \cdots$

(ii) $= \frac{1}{2}z^{-1} + [-z^{-1} - z^{-2} - z^{-3} - \cdots] + \left[-\frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \cdots\right]$

(ii) $= \frac{1}{2}z^{-1} + [-z^{-1} - z^{-2} - z^{-3} - \cdots] + \left[\frac{1}{2z} + \frac{1}{z^2} + \frac{2}{z^3} + \frac{4}{z^4}\cdots\right] = \frac{1}{z^3} + \frac{3}{z^4} + \frac{7}{z^5} + \frac{15}{z^6} + \cdots$

10. (a) There is one singularity at $z = 2$. It is a simple pole, so the residue is $\lim_{z\to 2}(z-2) \times 2/(2-z) = -2$.

(b) There are two singularities at $z = \pm 1$. Both are double. The residue at $z = 1$ is given by $\lim_{z\to 1} \frac{d}{dz}[(z-1)^2 \times e^z/(z^2-1)^2] = \lim_{z\to 1} \frac{d}{dz}[e^z/(z+1)^2] = \lim_{z\to 1}(z-1)e^z/(z+1)^3 = 0$. The residue at $z = -1$ is given by $\lim_{z\to -1} \frac{d}{dz}[(z+1)^2 \times e^z/(z^2-1)^2] = \lim_{z\to -1} \frac{d}{dz}[e^z/(z-1)^2] = \lim_{z\to -1}(z-3)e^z/(z-1)^3 = e^{-1}/2$.

(c) There is one singularity at $z = 0$. To find the residue we could either expand $\cos z$ in powers of z and find the appropriate term in the Laurent series directly, or we use the formula for a pole of order 5: The residue $= \lim_{z\to 0} \frac{1}{(5-1)!} \frac{d^4}{dz^4}[z^5 \times (\cos z)/z^5] = \frac{1}{24} \frac{d^4}{dz^4} \cos z|_{z=0} = \frac{1}{24}$.

(d) $\pi \tan \pi z$ has singularities at $z = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$. At a general singularity the residue is $\lim_{z\to 1/2+n\pi}(z - (1/2 + n\pi))\pi \tan \pi z = \lim_{z\to 1/2+n\pi} \frac{\pi(z-(1/2+n))}{\cot \pi z}$. Using l'Hospital's rule this gives a residue of $\lim_{z\to 1/2+n} \frac{\pi}{-\pi \csc^2 \pi z} = -1$ for all n.

11. (a) Using the standard substitution $z = e^{i\theta}$ we get

$$\int_0^{2\pi} \frac{1+\sin\theta}{2+\cos\theta}\, d\theta = \oint_C \frac{1+\frac{1}{2i}(z-z^{-1})}{2+\frac{1}{2}(z+z^{-1})} \frac{dz}{iz} = -\oint_C \frac{z^2+2iz-1}{z(z^2+4z+1)} dz.$$

The function inside the integral has singularities at $z = 0$ and $z = -2 \pm \sqrt{3}$. Only $z = 0$ and $z = -2 + \sqrt{3}$ lie inside the circle C of radius 1 centred on the origin. The residue at $z = 0$ is

$$\lim_{z\to 0} z \times \frac{z^2+2iz-1}{z(z^2+4z+1)} = \left.\frac{z^2+2iz-1}{z^2+4z+1}\right|_{z=0} = -1.$$

The residue at $z = -2 + \sqrt{3}$ is

$$\lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3}) \times \frac{z^2+2iz-1}{z(z^2+4z+1)} = \left.\frac{z^2+2iz-1}{z(z+2+\sqrt{3})}\right|_{z=-2+\sqrt{3}}$$

$$= \frac{6-4\sqrt{3}+2i(-2+\sqrt{3})}{(-2+\sqrt{3})2\sqrt{3}} = 1 + \frac{i}{\sqrt{3}}.$$

Hence the integral is $-2\pi i \times [(-1) + (1 + i/\sqrt{3})] = 2\pi/\sqrt{3}$.

(b) Again using the standard substitution $z = e^{i\theta}$ we get

$$\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta = \oint_C \frac{\frac{1}{2}(z^3+z^{-3})}{5+2(z+z^{-1})} \frac{dz}{iz}$$

$$= \frac{1}{4i} \oint_C \frac{z^6+1}{z^3(z^2+5/2z+1)} dz = \frac{1}{4i} \oint_C \frac{z^6+1}{z^3(z+2)(z+1/2)} dz,$$

where C is the circle of radius 1 centred on $z = 0$. This has a pole of order 3 at $z = 0$ and a simple pole at $z = 1/2$ that lie inside this circle. The residue at $z = -1/2$ is then

$$\lim_{z\to -1/2} (z+1/2) \times \frac{z^6+1}{z^3(z+2)(z+1/2)} = \frac{(-1/2)^6+1}{(-1/2)^3(-1/2+2)} = -65/12.$$

To find the residue at $z = 0$ we can simplify things by noting that if we split the expression $\frac{z^6+1}{z^3(z^2+5/2z+1)}$ into two parts, one with z^6 on the top, $\frac{z^6}{z^3(z^2+5/2z+1)} = \frac{z^3}{(z^2+5/2z+1)}$, and one with 1 on the top, $\frac{1}{z^3(z^2+5/2z+1)}$, we see that the former has no singularity at $z = 0$. This simplifies things a bit. The residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2} z^3 \times \frac{1}{z^3(z+2)(z+1/2)} = \lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{-2/3}{z+2} + \frac{2/3}{z+1/2}\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}\frac{2}{(z+2)^3} + \frac{2}{3}\frac{2}{(z+1/2)^3}\right]\Bigg|_{z=0} = -\frac{1}{12} + \frac{16}{3} = \frac{21}{4}.$$

Hence the integral is $2\pi i \times \frac{1}{4i} \times (-65/12 + 21/4) = -\pi/12$.

12. All these integrals are done by using the semicircular contour which has its diameter going between $-R$ and R along the real axis, and is closed in the upper half plane. In each case the contribution from the curved part of the contour vanishes as $R \to \infty$

(a) Poles at $\pm\frac{1+i}{\sqrt{2}}$ and $\pm\frac{-1+i}{\sqrt{2}}$, only the ones with a plus sign in front lie inside our contour. Residues at $\frac{1+i}{\sqrt{2}}$ and $\frac{1+i}{\sqrt{2}}$ are [using rule that if $q(z)$ has a simple zero at z_0, and $p(z)$ is non-zero at z_0 then the residue of $p(z)/q(z)$ is $P(z_0)/q'(z_0)$]

$$\frac{1+z^2}{4z^3}\Bigg|_{z=\frac{1+i}{\sqrt{2}}} = \frac{1}{2\sqrt{2}i} \qquad \text{and} \qquad \frac{1+z^2}{4z^3}\Bigg|_{z=\frac{-1+i}{\sqrt{2}}} = \frac{1}{2\sqrt{2}i}$$

respectively. Hence the integral around the contour is $2\pi i \times 2 \times \frac{1}{2\sqrt{2}i} = \sqrt{2}\pi$. But our integral only goes from 0 to ∞ and so the required integral is $\pi/\sqrt{2}$.

(b) Double poles at $+i$ and $-i$, only the former lies inside our contour.

The residue at $+i$ is

$$\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2 \times \frac{1}{(z^2+1)^2}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] = \frac{-2}{(z+i)^3}\Bigg|_{z=i} = \frac{1}{4i}.$$

Again our integral only goes from 0 to ∞ so the integral is $\frac{1}{2} \times 2\pi i \times \frac{1}{4i} = \pi/4$.

(c) Simple poles at $\pm i$ and $\pm 2i$, only those at i and $2i$ lie inside our contour.

The residues are

$$\lim_{z\to i}(z-i) \times \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z^2+4)}\Bigg|_{z=i} = \frac{1}{6i},$$

$$\lim_{z\to 2i}(z-2i) \times \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z^2+1)(z+2i)}\Bigg|_{z=2i} = -\frac{1}{12i},$$

respectively. This time the integral is from $-\infty$ to ∞, and so our answer is $2\pi i \times (\frac{1}{6i} - \frac{1}{12i}) = \pi/6$.

(d) The function inside the integral is an odd function, hence the integral is 0.