Mathematical Methods: Complex Variables 5

Answer Sheet

1. (a) $\lim_{n\to\infty}\frac{n+i}{2n-3} = 1/2 \neq 0$ so the series cannot converge.

   (b) Using the root test $\sqrt[n]{\left|\frac{(2i)^n n!}{n^n}\right|} = \frac{2}{n}\sqrt[n]{n!}$. You need to use Stirling's formula that $\sqrt[n]{n!} \approx n/e$ for large n to give $\sqrt[n]{|a_n|} \to 2/e < 1$, and so the series converges.

   If you use the ratio test you get $|a_{n+1}/a_n| = \left|2i\left(\frac{n}{n+1}\right)^n\right| = 2[(1+\frac{1}{n})^n]^{-1}$. Then we use $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ to give $|a_{n+1}/a_n| = 2/e < 1$ and so the series converges.

   (c) Using the ratio test $|a_{n+1}/a_n| = |\frac{1}{3}(z+2i)^2|$. This series will converge if this is less than 1, i.e. $|z+2i| < \sqrt{3}$, and diverge if it is greater than 1, i.e. $|z+2i| > \sqrt{3}$.

   (d) Using the ratio test $|a_{n+1}/a_n| = |n^2(z-i)/(n+1)|$. If $z \neq i$ then as $n \to \infty$ the ratio of successive terms tends to ∞, and so the series diverges. If $z = i$ then the series converges.

2. (a) Either find $f'(z) = \pi\cos\pi z$, $f''(z) = -\pi^2\sin\pi z$, $f'''(z) - \pi^3\cos\pi z$, $f''''(z) = \pi^4\sin\pi z$ etc. giving $f(1/2) = 1$, $f'(1/2) = 0$, $f''(1/2) = -\pi^2$, $f'''(1/2) = 0$, $f''''(1/2) = \pi^4$, etc. and so $f(z) = 1-\pi^2(z-\frac{1}{2})^2/2!+\pi^4(z-\frac{1}{2})^4/4!-\cdots$. Or simply use $\sin\pi z = \sin[\pi(z-\frac{1}{2})+\pi/2] = \sin\pi(z-\frac{1}{2})\cos\pi/2+\cos\pi(z-\frac{1}{2})\sin\pi/2 = \cos\pi(z-\frac{1}{2}) = 1-\pi^2(z-\frac{1}{2})^2/2!+\pi^4(z-\frac{1}{2})^4/4!-\cdots$.

   (b) Note error on question sheet, for $z_0 = 1$ read $z_0 = 0$.

$$\frac{e^z}{z-1} = -\left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots\right)\left(1+z+z^2+z^3+z^4+\cdots\right)$$

$$= -\left[1+(1+1)z+\left(1+1+\frac{1}{2!}\right)z^2+\left(1+1+\frac{1}{2!}+\frac{1}{3!}\right)z^3+\cdots+\left(\sum_{m=0}^{n}\frac{1}{m!}\right)z^n+\cdots\right]$$

3. (a) Use partial fractions:

$$\frac{2}{1-z^2} = \frac{1}{1-z}+\frac{1}{1+z} = -\frac{1}{z-1}+\frac{1}{2+(z-1)}$$

   If $|z-1| < 2$ then

$$\frac{2}{1-z^2} = -\frac{1}{z-1}+\frac{1}{2}\left(\frac{1}{1+\frac{z-1}{2}}\right)$$

$$= -\frac{1}{(z-1)}+\frac{1}{2}-\frac{(z-1)}{4}+\frac{(z-1)^2}{8}-\frac{(z-1)^3}{16}+\cdots$$

   If $|z-1| > 2$ then

$$\frac{2}{1-z^2} = -\frac{1}{z-1}+\frac{1}{(z-1)}\left(\frac{1}{1+\frac{2}{z-1}}\right)$$

$$= -\frac{1}{(z-1)}+\frac{1}{(z-1)}\left[1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}-\frac{8}{(z-1)^3}+\cdots\right]$$

$$= -\frac{2}{(z-1)^2}+\frac{4}{(z-1)^3}-\frac{8}{(z-1)^4}+\cdots$$

(b) Do some simple manipulations first:

$$\frac{e^z}{z-1} = \frac{e^{z-1+1}}{z-1} = e \times \frac{e^{z-1}}{z-1} = \frac{e}{(z-1)}\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right]$$

$$= \frac{e}{(z-1)} + e + \frac{e}{2!}(z-1) + \frac{e}{3!}(z-1)^2 + \cdots$$

4. (a) Singularities at $z = 0$ and $z = 1$

$$\operatorname*{Res}_{z=0}\left[\frac{z^2+1}{z^2-z}\right] = \lim_{z\to 0}\frac{z^2+1}{z-1} = -1.$$

$$\operatorname*{Res}_{z=1}\left[\frac{z^2+1}{z^2-z}\right] = \lim_{z\to 0}\frac{z^2+1}{z} = 2.$$

(b) There is only one singularity, a pole of order 5, at $z = -1$.

$$\operatorname*{Res}_{z=-1}\left[\frac{e^z}{(z+1)^5}\right] = \lim_{z\to -1}\left[\frac{1}{4!}\frac{d^4}{dz^4}\left((z+1)^5 \times \frac{e^5}{(z+1)^5}\right)\right] = \frac{1}{4!}\left.\frac{d^4 e^z}{dz^4}\right|_{z=-1} = \frac{e^{-1}}{24}$$

(c) There are singularities at $z = 0$, $z = -1/2$ and $z = -2$. The first of these is of order 3, while the other two are simple poles. Note

$$\operatorname*{Res}_{z=0}\left[\frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \operatorname*{Res}_{z=0}\left[\frac{z^6}{z^3(z+2)(z+\frac{1}{2})}\right] - \operatorname*{Res}_{z=0}\left[\frac{1}{z^3(z+2)(z+\frac{1}{2})}\right]$$

But the first of the two residues of the right has a z^6 term in the top which more than cancels out the z^3 in the denominator, and so the residue from this term is 0, i.e.

$$\operatorname*{Res}_{z=0}\left[\frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \operatorname*{Res}_{z=0}\left[\frac{-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \lim_{z\to 0}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left(z^3 \times \frac{-1}{z^3(z+2)(z+\frac{1}{2})}\right)\right]$$

$$= -\frac{1}{2}\frac{d^2}{dz^2}\left.\left[\frac{1}{(z+2)(z+\frac{1}{2})}\right]\right|_{z=0} = -\frac{1}{2}\frac{d^2}{dz^2}\left.\left[\frac{-\frac{2}{3}}{(z+2)} + \frac{\frac{2}{3}}{(z+\frac{1}{2})}\right]\right|_{z=0}$$

$$= -\frac{1}{2}\left.\left[\frac{-\frac{4}{3}}{(z+2)^3} + \frac{\frac{4}{3}}{(z+\frac{1}{2})^3}\right]\right|_{z=0} = -\frac{1}{2} \times \frac{4}{3}\left[-\frac{1}{8} + 8\right] = -\frac{21}{4}$$

$$\operatorname*{Res}_{z=-2}\left[\frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \lim_{z\to -2}\left[(z+2) \times \frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \left.\frac{z^6-1}{z^3(z+\frac{1}{2})}\right|_{z=-2} = \frac{21}{4}$$

$$\operatorname*{Res}_{z=-\frac{1}{2}}\left[\frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \lim_{z\to -\frac{1}{2}}\left[(z+\tfrac{1}{2}) \times \frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}\right] = \left.\frac{z^6-1}{z^3(z+2)}\right|_{z=-\frac{1}{2}} = \frac{21}{4}$$

(d) There are simple poles at all integer values of z, i.e. $z = \ldots, -2, -1, 0, 1, 2, \ldots$. It is easiest to use the rule for finding residues of ratios of two functions $p(z)/q(z)$ at a simple zero of $q(z)$, say z_0. At this point the residue is $p(z_0)/q'(z_0)$ provided $p(z_0) \neq 0$.

$$\operatorname*{Res}_{z=n}[\pi \cot \pi z] = \operatorname*{Res}_{z=n}\left[\frac{\pi\cos\pi z}{\sin\pi z}\right] = \left.\frac{\pi\cos\pi z}{\frac{d}{dz}\sin\pi z}\right|_{z=n} = \left.\frac{\pi\cos\pi z}{\pi\cos\pi z}\right|_{z=n} = 1$$

5. The choice of integrals here wasn't the most inspired, sorry. In both cases use the standard substitution $z = e^{i\theta}$ to transform the integrals into integrals around a circle of radius 1 centred on the origin in the z-plane.

(*a*)
$$\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\,d\theta = \oint \frac{\frac{1}{2i}(z-z^{-1})}{3+\frac{1}{2}(z+z^{-1})}\,\frac{dz}{iz} = -\oint \frac{z^2-1}{z(z^2+6z+1)}\,dz$$

The integrand has singularities at $z = 0$, $z = -3+\sqrt{8}$ and $z = -3-\sqrt{8}$. Only the first two of these lie inside the circle. We need to find their residues:

$$\operatorname*{Res}_{z=0}\left[\frac{z^2-1}{z(z^2+6z+1)}\right] = \lim_{z\to 0}\left[z \times \frac{z^2-1}{z(z^6+1)}\right] = \left.\frac{z^2-1}{z^2+6z+1}\right|_{z=0} = -1.$$

$$\operatorname*{Res}_{z=-3+\sqrt{8}}\left[\frac{z^2-1}{z(z^2+6z+1)}\right] = \lim_{z\to-3+\sqrt{8}}\left[(z+3-\sqrt{8}) \times \frac{z^2-1}{z(z^2+6z+1)}\right]$$
$$= \left.\frac{z^2-1}{z(z+3+\sqrt{8})}\right|_{z=-3+\sqrt{8}} = \frac{16-6\sqrt{8}}{2\sqrt{8}(-3+\sqrt{8})} = 1.$$

The two residues cancel each other out to give the integral as 0.

(*b*)
$$\int_0^{2\pi} \frac{\sin 3\theta}{5+4\cos\theta}\,d\theta = \oint \frac{\frac{1}{2i}(z^3-z^{-3})}{5+2(z+z^{-1})}\,\frac{dz}{iz} = -\frac{1}{4}\oint \frac{z^6-1}{z^3(z^2+\frac{5}{2}z+1)}\,dz$$

Since $z^2+\frac{5}{2}z+1 = (z+2)(z+\frac{1}{2}$ we can now use the results from 4(c). The integrand has residues at $z = 0$, $z = -\frac{1}{2}$ and $z = -2$. The first two of which lie inside the unit circle. The residues from these points are $-\frac{21}{4}$ and $\frac{21}{4}$ respectively. These again cancel out to give the answer that the integral is 0.

6. In both these cases use the semicircular contour, C, closed in the upper-half plane. The contributions from the curved arc is zero in both cases in the limit as the radius tends to ∞.

(*a*)
$$\int_0^{\infty} \frac{dx}{1+4x^2} = \frac{1}{2}\int_C \frac{dz}{1+4z^2}$$

There are poles at $z = \pm\frac{i}{2}$, only the one at $z = \frac{i}{2}$ lies inside the contour.

$$\operatorname*{Res}_{z=\frac{i}{2}} \frac{1}{1+4z^2} = \left.\frac{1}{8z}\right|_{z=\frac{i}{2}} = -\frac{i}{4}$$

Hence
$$\int_0^{\infty} \frac{dx}{1+4x^2} = \frac{1}{2} \times 2\pi i \times -\frac{i}{4} = \frac{\pi}{4}$$

(*b*)
$$\int_{-\infty}^{\infty} \frac{1+2x^2}{x^4+10x^2+9}\,dx = \int_C \frac{1+2z^2}{(z^2+9)(z^2+1)}\,dz.$$

The integrand has 4 poles at $z = \pm 3i$ and $z = \pm i$, only the ones at $z = 3i$ and $z = i$ lie inside the contour.

$$\operatorname*{Res}_{z=i} \frac{1+2z^2}{(z^2+1)(z^2+9)} = \left.\frac{1+2z^2}{(z+i)(z^2+9)}\right|_{z=i} = \frac{1-2}{2i(-1+9)} = \frac{i}{16},$$

$$\operatorname*{Res}_{z=3i} \frac{1+2z^2}{(z^2+1)(z^2+9)} = \left.\frac{1+2z^2}{(z^2+1)(z+3i)}\right|_{z=3i} = \frac{1-18}{(-9+1)(3i+3i)} = -\frac{17i}{48}.$$

Hence the integral is

$$\int_{-\infty}^{\infty} \frac{1+2x^2}{x^4+10x^2+9}\,dx = 2\pi i \times \left(\frac{i}{16} - \frac{17i}{48}\right) = \frac{7\pi}{12}.$$

7. (a) We again use the semicircular contour closed in the upper half plane. However instead of considering the integral of $\frac{\cos az}{b^2+z^2}$ we consider $\frac{e^{iaz}}{b^2+z^2}$. The reason for this is that $\cos az$ grows exponentially as the magnitude of the imaginary part of az increases. This means that the contribution from the curved part of the contour will not be negligible. If we use instead e^{iaz} then the real part of this along the real axis is just the $\cos ax$ we want, and (provided $a > 0$) it decays exponentially away from the real axis in the upper half plane:

$$\int_{-\infty}^{\infty} \frac{\cos ax}{b^2+x^2}\,dx = \operatorname{Re} \int_C \frac{e^{iaz}}{b^2+z^2}\,dz.$$

There is one pole inside the contour at $z = ib$, where

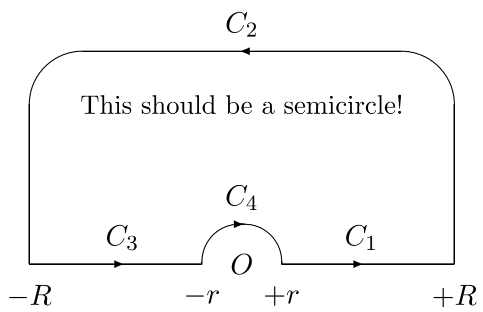
$$\operatorname*{Res}_{z=ib} \left[\frac{e^{iaz}}{b^2+z^2}\right] = \left.\frac{e^{iaz}}{z+ib}\right|_{z=ib} = \frac{1}{2ib}e^{-ab}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos ax}{b^2+x^2}\,dx = \operatorname{Re}\left[2\pi i \times \frac{e^{-ab}}{2ib}\right] = \frac{\pi e^{-ab}}{b}.$$

(b) This one is more complicated as we have to be careful with the origin. For the same reason as in part (a) we do not use $\sin z$ in our contour integration, but use e^{iz} in order that the integrand decays in the upper-half plane. This means that what would at first appear to be a removable singularity at $z = 0$ is now a simple pole. However it lies directly on our intended path of integration! So we have to take a detour around it. This could be done in either the upper-half plane or the lower-half plane.

We choose the upper half plane as it means we don't have to worry about the residue at this point as the singularity is not enclosed by the contour. The contour is shown to the right.



$$\int_{C_1} \frac{e^{iz}}{z}\,dz = \int_r^R \frac{\cos x}{x}\,dx + i\int_r^R \frac{\sin x}{x}\,dx$$

$$\int_{C_3} \frac{e^{iz}}{z}\,dz = \int_{-R}^{-r} \frac{\cos x}{x}\,dx + i\int_{-R}^{-r} \frac{\sin x}{x}\,dx = -\int_r^R \frac{\cos x}{x}\,dx + i\int_r^R \frac{\sin x}{x}\,dx$$

Adding these together gives

$$\int_{C_1} \frac{e^{iz}}{z}\,dz + \int_{C_3} \frac{e^{iz}}{z}\,dz = 2i\int_r^R \frac{\sin x}{x}\,dx$$

This removes problems associated with $\cos x/x \to \infty$ as $x \to 0$.

$$\int_{C_2} \frac{e^{iz}}{z}\,dz = \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\,iRe^{i\theta}d\theta = \int_0^\pi ie^{iRe^{i\theta}}\,d\theta$$
$$= \int_0^\pi ie^{iR\cos\theta - R\sin\theta}\,d\theta.$$

This last integral does not vanish along the path of integration ar $R \to \infty$, we have to be more careful and use the method used on the handout for finding the integrals $\int_0^\infty \cos x^2\,dx$ and $\int_0^\infty \sin x^2\,dx$. Note that

$$\left|\int_0^\pi ie^{iR\cos\theta - R\sin\theta}\,d\theta\right| \le \int_0^\pi \left|ie^{iR\cos\theta - R\sin\theta}\right|\,d\theta = \int_0^\pi e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta.$$

We then use the result that, in the range $0 \le \theta \le \pi/2$, $\sin\theta \ge 2\theta/\pi$. Hence $e^{-R\sin\theta} \le e^{-2R\theta/\pi}$. This givesin the limit $R \to \infty$

$$\left|\int_0^\pi ie^{iR\cos\theta - R\sin\theta}\,d\theta\right| \le 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \left[\frac{\pi}{R}e^{-2R\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{R}\left(1 - e^{-R\pi^2}\right) \to 0$$

Lastly

$$\int_{C_4} \frac{e^{iz}}{z}\,dz = \int_\pi^0 \frac{e^{ire^{i\theta}}}{re^{i\theta}}\,ire^{i\theta}d\theta = \int_\pi^0 ie^{ire^{i\theta}}\,d\theta.$$

As $\theta \to 0$ this integral tends to $-i\pi$. Combining all these results in the limit $R \to \infty$ and $r \to 0$ gives

$$0 = \left(\int_{C_1} + \int_{C_3}\right) + \int_{C_2} + \int_{C_4} = 2i\int_0^\infty \frac{\sin x}{x}\,dx + 0 - i\pi$$

Giving

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

8. With the Laplace transform defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt,$$

we can find the Laplace transform of the derivative of a function:

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt = -f(0) + s\mathcal{L}(f).$$

Similarly

$$\mathcal{L}(f'') = -sf(0) - f'(0) + s^2\mathcal{L}(f).$$

In our case $f(0) = 0$ and $f'(0) = 1$. By taking the Laplace transform of our equation we get

$$-1 - 0 + s^2F(s) - 4(-0 + sF(s)) + 3F(s) = 2\mathcal{L}(t) + \mathcal{L}(1) = \frac{2}{s^2} + \frac{1}{s}.$$

Hence

$$F(s) = \frac{1}{s^2 - 4s + 3}\left(\frac{2}{s^2} + \frac{1}{s} + 1\right) = \frac{s^2 + s + 2}{s^2(s^2 - 4s + 3)}.$$

To invert this transform we use the Bromwich integral

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds,$$

where γ is chosen so that the path of integration lies to the right of all the singularities. In this case the singularities of $F(s)$ are at $s = 0$, 1 and 3. We will close the contour in the left half plane, where the integrand decays sufficiently rapidly for there to be no contribution from that part of the contour. To calculate the contour integral we find the residues of $e^{st}F(s)$ at the singularities.

$$\operatorname*{Res}_{s=3}\left[\frac{e^{st}(s^2 + s + 2)}{s^2(s^2 - 4s + 3)}\right] = \left.\frac{e^{st}(s^2 + s + 2)}{s^2(s - 1)}\right|_{s=3} = \frac{7e^{3t}}{9},$$

$$\operatorname*{Res}_{s=1}\left[\frac{e^{st}(s^2 + s + 2)}{s^2(s^2 - 4s + 3)}\right] = \left.\frac{e^{st}(s^2 + s + 2)}{s^2(s - 3)}\right|_{s=1} = -2e^t,$$

$$\operatorname*{Res}_{s=0}\left[\frac{e^{st}(s^2 + s + 2)}{s^2(s^2 - 4s + 3)}\right] = \lim_{s\to 0}\left[\frac{d}{ds}\left(s^2 \times \frac{e^{st}(s^2 + s + 2)}{s^2(s^2 - 4s + 3)}\right)\right] = \frac{d}{ds}\left.\left(\frac{e^{st}(s^2 + s + 2)}{(s^2 - 4s + 3)}\right)\right|_{s=0}$$

$$= \left.\frac{te^{st}(s^2 + s + 2)}{(s^2 - 4s + 3)} + \frac{e^{st}(2s + 1)}{(s^2 - 4s + 3)} - \frac{e^{st}(s^2 + s + 2)(2s - 4)}{(s^2 - 4s + 3)^2}\right|_{s=0}$$

$$= \frac{t \times 2}{3} + \frac{1}{3} - \frac{2 \times -4}{3^2} = \frac{2t}{3} + \frac{11}{9}$$

Hence

$$f(t) = \frac{1}{2\pi i} \times 2\pi i\left(\frac{7e^{3t}}{9} - 2e^t + \frac{2t}{3} + \frac{11}{9}\right).$$

We can check this is the solution:

$$f'(t) = \frac{7e^{3t}}{3} - 2e^t + \frac{2}{3} \qquad \text{and} \qquad f''(t) = 7e^{3t} - 2e^t$$

giving $f'(0) = \frac{7}{3} - 2 + \frac{2}{3} = 1$ and $f(0) = \frac{7}{9} - 2 + \frac{11}{9} = 0$ which are the correct initial conditions, and

$$f''(t) - 4f'(t) + 3f(t) = \left(7 - 4 \times \frac{7}{3} + 3 \times \frac{7}{9}\right)e^{3t} + (-2 - 4 \times (-2) + 3 \times (-2))e^t$$

$$+ \left(-4 \times \frac{2}{3} + 3 \times \frac{11}{9}\right) + 3 \times \frac{2t}{3} = 2t + 1$$

as required.