Mathematical Methods: Complex Variables 1

1. Perform the following operations analytically and where appropriate show graphically:

   (a) $(2+3i)+(1+4i)$, (b) $(6-2i)+(-3+4i)$,

   (c) $(3+2i)-(-2+i)$, (d) $(2+3i)\times(1-2i)$,

   (e) $(2+4i)\times(-1+2i)$, (f) $\dfrac{i(2+3i)}{1-2i}$.

2. Find

   (a) $\mathrm{Re}\left(\dfrac{1}{1+i}\right)$, (b) $\mathrm{Im}\,\dfrac{(2-3i)^2}{2+3i}$,

   (c) $\mathrm{Im}\,\dfrac{z}{\overline{z}}$, (d) $\mathrm{Re}\,(a+ib)^n$ with a, b real.

3. Show that if $z_1 z_2 = 0$ then at least one of z_1 and z_2 is zero.

4. Represent in polar form

   (a) $1+i$, (b) $\dfrac{1+i}{1-i}$,

   (c) $\dfrac{9+13i}{1+3i}$, (d) $\dfrac{3\sqrt{2}+2i}{-\sqrt{2}-2i/3}$.

   (e) $\dfrac{(1+i)(2+i)}{3-i}$, (f) $\dfrac{1-\cos\phi-i\sin\phi}{1+\cos\phi+i\sin\phi}$.

5. Verify $r(\cos\theta+i\sin\theta)\times r'(\cos\theta'+i\sin\theta') = rr'(\cos(\theta+\theta')+i\sin(\theta+\theta'))$.

   Use this result to demonstrate the formula of De Moivre:

   $$(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

   Hence find an expression for $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

6. Find all five solutions to $z^5=1$, and show them graphically.

7. Demonstrate that $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$.

8. Show that $|z_1+z_2|^2+|z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right)$ (the parallelogram equality).

9. What part of the complex plane is represented by the inequality

   $$|z+1| > |z-1|$$

   Give both an analytical proof and a geometrical argument.

10. Show that the two following statements are equivalent

$$\left|\frac{a-b}{a-\overline{b}}\right| < 1 \qquad \text{and} \qquad \text{Im}a \times \text{Im}b > 0.$$

11. Show that the line $ax + by = c$ where a, b, c are all reals can be expressed as $B\overline{z} + \overline{B}z = c$ for some complex B (to be found).

12. Show that a circle of radius r and centre a (complex) can be described by the formula $z\overline{z} + B\overline{z} + \overline{B}z + C = 0$, where C is real, and B is complex. What value does C take when the circle passes through the origin.

13. Consider the map $z \to \frac{1}{z}$. By using the results of the previous two questions, show that in general this maps straight lines onto circles and circles onto circles. In what special cases does it map circles onto straight lines and straight lines onto straight lines?

14. Find the real and imaginary parts of the following functions and evaluate them at $z = 1 + i$:

    (a) $z/(z-1)$ (b) $z^2 + 3iz - 3$

    (c) z^4 (d) $1/(z+1)^2$

15. Show that (a) $\overline{z}$, (b) $|z|$ and (c) z^2 are continuous in the region $|z| < 1$, and secondly that they are uniformly continuous in this same region.

16. Especially for those that think they are very good at integration. Evaluate, without using any complex variable theory,

    (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$, (b) $\displaystyle\int_0^{\infty} \frac{dx}{x^6+1}$,

    (c) $\displaystyle\int_0^{\infty} \sin x^2 \, dx$.