Mathematical Methods: Complex Variables 2

1. If $f(z)$ and $g(z)$ are both analytic functions show that (a) $f(z) + g(z)$ and (b) $f(g(z))$ are analytic functions.

2. Use the Cauchy-Riemann equations to decide whether the following functions are analytic:

   (a) $f(z) = z^3$ (b) $f(z) = iz\overline{z}$

   (c) $f(z) = \arg z$ (d) $f(z) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$

3. Show that if an analytic function, $f(z)$, has constant modulus for all z, then $f(z)$ is a constant. [Hint: consider the variation of $|f(z)|^2 = u(x,y)^2 + v(x,y)^2$, and use the Cauchy-Riemann equations.]

4. Show that is $f(z)$ is an analytic function then so is $\overline{f(\overline{z})}$, denoted $\overline{f}(z)$. [e.g. If $f(z) = z^2 + 3iz + 4 + i$, then $\overline{f}(z) = \overline{f(\overline{z})} = \overline{\overline{z}^2 + 3i\overline{z} + 4 + i} = z^2 - 3iz + 4 - i$.]

5. Show that the function $f(z) = u(x,y) + iv(x,y)$ where

   $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0, \quad \text{and} \quad f(0) = 0$$

   is continuous, and that the Cauchy-Riemann equations are satisfied at the origin. Then show that $f'(0)$ does not exist.

6. Show the following functions are harmonic, and find the conjugate harmonic functions. Then use the four methods for constructing the analytic function from these functions, timing each one in turn. Which was the fastest method?

   (a) $y^2 - x^2$ (b) $e^x \cos y$

7. [Question 6(a) from 1992 exam] Show that $u(x,y) = x^3 - 3xy^2 - 2x$ is harmonic and find a corresponding analytic function $w = f(z)$ of which $u(x,y)$ is the real part.

   Show that families of curves in the z plane, which correspond to constant values of u and its conjugate harmonic v in the w plane, are orthogonal.

8. [Question 5(a) from 1993 exam] Show that $u(x,y) = x + \sin x \cosh y$ is harmonic and find a corresponding analytic function $w = f(z)$ of which $u(x,y)$ is the real part.

   Show that families of curves in the z plane, which correspond to constant values of u and its conjugate harmonic v in the w plane, are orthogonal.

9. Show that $1/z$ is analytic for all $z \neq 0$. Given that all polynomials are analytic, and that polynomials $p(z)$ and $q(z)$ have no common zeros, show that the rational function

$$f(z) = \frac{p(z)}{q(z)}$$

is also analytic except at the zeros of $q(z)$.

10. Find all possible values of z for which (a) $\cos z$ and (b) $\sin z$ have real values.

11. Derive the rules

$$\cos^2 z + \sin^2 z = 1,$$

$$\cos(-z) = \cos z, \quad \sin(-z) = -\sin z, \quad \tan(-z) = -\tan z,$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2,$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1.$$

12. Find all possible values of i^i.