Mathematical Methods: Complex Variables 4

1. Prove the Fundamental Theorem of Algebra, that a polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where the degree $n \geq 1$ and $a_n \neq 0$, has at least one root. [Hint: Assume it has no roots, consider $1/P(z)$ and apply Liouville's Theorem.]

   Further show that if z_1 is a root show that $P(z) = (z - z_1)Q(z)$ where $Q(z)$ is a polynomial of degree $n-1$. Hence show that $P(z) = a_n(z-z_1)(z-z_2)\ldots(z-z_n)$ and so has n roots. [Hint: Consider $P(z) = P(z) - P(z_1)$]

2. (a) Let α be an isolated singularity of the analytic function f. Define the statements:

      i. α is a removable singularity,

      ii. α is a pole,

      iii. α is an essential singularity.

   (b) Assume that α is also an isolated singularity of $1/f(z)$. In each of the above cases describe the nature of the singularity of $1/f(z)$ at α with proof.

   (c) Find the singularities of the function

   $$f(z) = \frac{e^{1/z}(z - \pi)}{\sin z}$$

   and the nature of each singularity.

3. Show that the *sequence* $a_n = \sqrt{n+1} - \sqrt{n}$ converges, but that the *series* $\sum a_n$ diverges.

4. Which of the following series converge, giving your reasons

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{3n+2i}$ (b) $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{1+i}{2}\right)^n$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{n!}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$ [You may not find this one obvious]

5. *In lectures we gave the method for finding the radius of convergence of the power series $\sum_0^\infty a_n(z-z_0)^n$ by looking at the limit of $|a_{n+1}/a_n|$ as $n \to \infty$. This method was based on the ratio test. Another method is based on the root test:*

   Show that if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then the power series has a radius of convergence of $R = 1/L$.

   This approach has the advantage that it can be used to provide the radius of convergence for "badly behaved" series.

   If we consider the *sequence* $\{\sqrt[n]{|a_n|}\}$ then if it remains bounded it will have some limit points (see first handout), i.e. points in neighbourhoods of which there are an infinite number of points of the sequence. If L^* is the greatest such limit point then show that the radius of convergence of the power series is $R = 1/L^*$. (Don't spend too much time on this if you can't see how it is done.)

Consider the power series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n = 3+2^{-1}z+(2+2^{-2})z^2+2^{-3}z^3+(2+2^{-4})z^4+\cdots$$

Find the limits of $\sqrt[n]{|a_n|}$ for even n and for odd n as $n \to \infty$. Which of these is the larger? What is the radius of convergence of the series.

When applying the above and the root test you may find it useful to know that for large n, $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ (Stirling's formula), and so $\sqrt[n]{n!} \approx n/e$ (I do not expect you to remember this!)

6. Find the Taylor series for (a) $\log(z+1)$, (b) $1/(z+1)$ and (c) $-1/(z+1)^2$ about $z = 0$. Verify that you can obtain series (a) by term-by-term integration of series (b), and series (c) by term-by-term differentiation of series (b).

7. Find the Taylor series for the following functions about the centres z_0, giving the radii of convergence

(a) $\dfrac{1}{1+z}$, $\quad z_0 = 1$ (b) $\dfrac{1}{1+z^2}$, $\quad z_0 = 1$

(c) $\sin \pi z^2$, $\quad z_0 = 2$ (d) $\ln 2z$, $\quad z_0 = 2$

8. Find the coefficients of the Laurent series for $1/(1-z)$ about the centre $z = -1$ using the integral formulæ for the coefficients a_n and b_n, showing clearly the different regions of validity for the different series.

9. Find all the Laurent series for the following functions about the given centres z_0

(a) $\dfrac{1}{z-3}$, $\quad z_0 = 0$ (b) $\dfrac{1}{z(z-1)(z-2)}$, $\quad z_0 = 0$

10. Find *all* the singularities and calculate the corresponding residues for the following:

(a) $\dfrac{2}{2-z}$ (b) $\dfrac{e^z}{(z^2-1)^2}$

(c) $\dfrac{\cos z}{z^5}$ (d) $\pi \tan \pi z$

11. Evaluate the integrals

(a) $\displaystyle\int_0^{2\pi} \frac{1+\sin\theta}{2+\cos\theta}\,d\theta$ (b) $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\,d\theta$

12. Evaluate the integrals

(a) $\displaystyle\int_0^{\infty} \frac{1+x^2}{1+x^4}\,dx$ (b) $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}\,dx$ (d) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3}{1+x^6}\,dx$ [Think first!]