Mathematical Methods: Complex Variables 5

Coursework Questions. To be handed in Monday, 25 April

1. Investigate the convergence or divergence of the following series and power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+i}{2n-3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2i)^n n!}{n^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(z+2i)^{2n}}{3^n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2}(z-i)^n$

2. Find the Taylor series for the following functions about the given centres, z_0:

   (a) $\sin \pi z, \qquad z_0 = \frac{1}{2}$ (b) $\displaystyle\frac{e^z}{z-1}, \qquad z_0 = 1$

3. Find the Laurent series for the following functions about the given centres, z_0:

   (a) $\displaystyle\frac{2}{1-z^2}, \qquad z_0 = 1$ (b) $\displaystyle\frac{e^z}{z-1}, \qquad z_0 = 1$

4. Find *all* the singularities and the corresponding residues of the following functions:

   (a) $\displaystyle\frac{z^2+1}{z^2-z}$ (b) $\displaystyle\frac{e^z}{(z+1)^5}$

   (c) $\displaystyle\frac{z^6-1}{z^3(z+2)(z+\frac{1}{2})}$ (d) $\pi \cot \pi z$

5. Evaluate the integrals using contour integration techniques

   (a) $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\, d\theta$ (b) $\displaystyle\int_0^{2\pi} \frac{\sin 3\theta}{5+4\cos\theta}\, d\theta$

6. Evaluate the integrals using contour integration techniques

   (a) $\displaystyle\int_0^{\infty} \frac{dx}{1+4x^2}$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{1+2x^2}{x^4+10x^2+9}\, dx$

7. Evaluate the following integrals:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{b^2+x^2}\, dx$

   [Hint: Consider $\cos ax$ to be the real part of e^{iaz}, then choose as your contour of integration the semicircle radius R in the upper-half plane (why?) centred on 0 with its base running along the real axis as shown.]

   (b) $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$

[Hint: Remember $\sin x$ is the imaginary part of e^{iz} when z lies on the real axis. Use the contour from the previous integral, but with a small semicircular detour of radius r around the origin as shown.]

8. The function $f(t)$ is a solution to the initial value problem

$$f'' - 4f' + 3f = 2t + 1, \qquad f(0) = 0, \quad f'(0) = 1.$$

Find the Laplace transform $F(s)$ of $f(t)$.

Using the inverse Laplace transform

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds$$

find the solution $f(t)$. What are the restrictions on the value of γ that you can choose? Verify that your solution satisfies the original differential equation and the initial conditions.