Models 5

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for the London Taught Course Centre

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Dynamical Systems

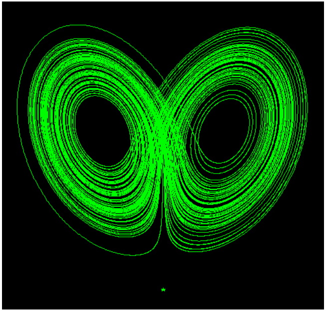
Dynamical systems theory has had a significant impact on applied mathematics and models.

We will briefly look at two dynamical systems:

- Thermal convection — Lorenz system
- Earth's magnetic field — Rikitake model

The Lorenz Attractor

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = \rho x - xz - y$$
$$\frac{dz}{dt} = xy - \beta z$$

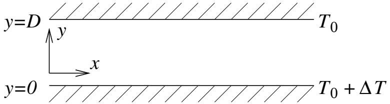


Usually with $\sigma = 10$ or 7, $\beta = 8/3$ and with ρ varied.
Where did this come from?

Lorenz was originally studying a model of convection in the atmosphere.

- He observed irregular behaviour — chaos.
- When he tried to reproduce the results by re-entering numbers from a previous run he got behaviour that diverged — sensitivity on initial conditions.

Convection in a layer of fluid



$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla p + g\alpha T \hat{\mathbf{y}} + \nu \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T$$

Here we make the Boussinesq approximation — density variations neglected except in the buoyancy term.
Density given by

$$\rho = \rho_0(1 - \alpha(T - T_0))$$

The boundary conditions are

$$T = T_0, \quad v = 0 \quad \text{on} \quad y = D$$

$$T = T_0 + \Delta T, \quad v = 0 \quad \text{on} \quad y = 0$$

In reality we should have no-slip boundary conditions $u = 0$ on $y = 0$ and $y = D$, but often stress-free boundary conditions are used, which makes analysis easier (as we will do here).

Dimensional quantities

$$\nu \sim L^2 T^{-1}, \qquad \kappa \sim L^2 T^{-1}, \qquad g \sim L T^{-2},$$

$$\rho_0 \sim M L^{-3}, \qquad \Delta T \sim \theta, \qquad \alpha \sim \theta^{-1}, \qquad D \sim L.$$

Seven quantities involving in 4 basic quantities, so would have 3 non-dimensional equations. But observe that g only appears multiplied by α we have another restriction that reduces things to two quantities.

Non-dimensionalising the equations using scalings length $\sim D$, time$\sim D^2/\kappa$, temperature$\sim \Delta T$, speed$\sim \kappa/D$, pressure$\sim \rho_0\kappa\nu/D^2$, we obtain

$$\frac{1}{\sigma}\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right] = -\nabla p + \mathrm{Ra}\, T\hat{\mathbf{y}} + \nabla^2\mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T$$

where

$$\mathrm{Ra} = \frac{g\alpha\Delta T D^3}{\nu\kappa} \qquad \sigma = \frac{\nu}{\kappa}$$

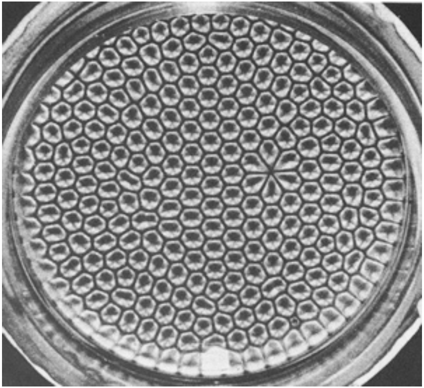
These have the simple solution

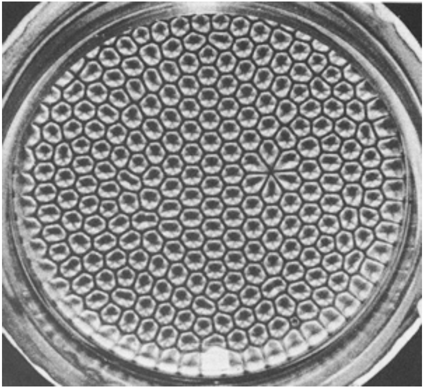
$$\mathbf{u} = \mathbf{0}, \qquad T = 1 - y$$

— the conduction solution.

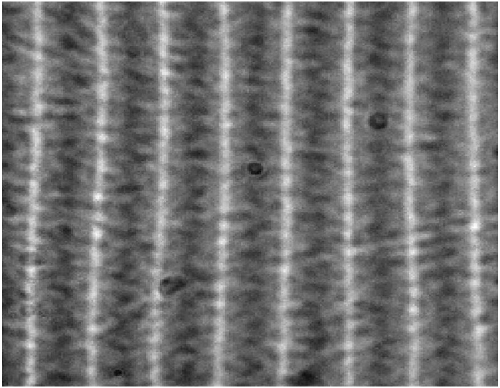
Instabilities

When the temperature difference is small this is the solution that is observed. When sufficiently large convection starts — the fluid starts moving.





A classic experiment, and much reproduced photograph by Koschmieder (1974) — but not really the problem under consideration.



It can be shown that between parallel plates the onset is usually in the form of parallel convection rolls.

Look at a 2-dimensional version perpendicular to the roll axes.

If we use a streamfunction ψ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial y}$$

and if we take T to be the deviation from the conduction solution then the governing equations become (after taking the curl of the momentum equation)

$$\frac{1}{\sigma}\left[\frac{\partial \nabla^2 \psi}{\partial t} + \psi_y \nabla^2 \psi_x - \psi_x \nabla^2 \psi_y\right] = \mathrm{Ra}\, T_x - \nabla^4 \psi$$

$$\frac{\partial T}{\partial t} + \psi_y T_x - \psi_x T_y + \psi_x = \nabla^2 T$$

Linear stability

Firstly we will look at the linear stability — we neglect the nonlinear terms.

We can look for solutions of the form

$$\psi = \psi(y) \sin \alpha x \, e^{\lambda t}, \qquad T = T(y) \cos \alpha x \, e^{\lambda t}.$$

If we use realistic boundary conditions we end up with an eigenvalue problem involving a 4^{th}-order ODE for $\psi(y)$ and a 2^{nd}-order ODE for $T(y)$.

If we use stress-free boundary conditions we find both ψ and T have a y-dependency of $\sin n\pi y$, for $n = 1, 2, \ldots$

The eigenvalue problem then becomes

$$\left[\frac{\lambda}{\sigma} + (\alpha^2 + n^2\pi^2)\right] \left[\lambda + (\alpha^2 + n^2\pi^2)\right] (\alpha^2 + n^2\pi^2) = \alpha^2 \mathrm{Ra}.$$

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For steady onset of convection (as opposed to oscillatory onset) $\lambda = 0$ and

$$\mathrm{Ra} = \frac{(\alpha^2 + n^2\pi^2)^3}{\alpha^2}$$

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This has a minimum when $n = 1$, $\alpha = \pi/\sqrt{2}$ and

$$\mathrm{Ra} = \frac{27\pi^4}{4}$$

Note:

- It can be shown by the use of **Energy stability analysis** that any nonlinear disturbance will decay for values of Ra less than the critical value found here (D. D. Joseph, 1965).

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- The original experiments had a free surface, and variations in the temperature at the surface lead to variations in the surface tension which drives the flows — Bénard-Marangoni convection.

Weakly nonlinear analysis

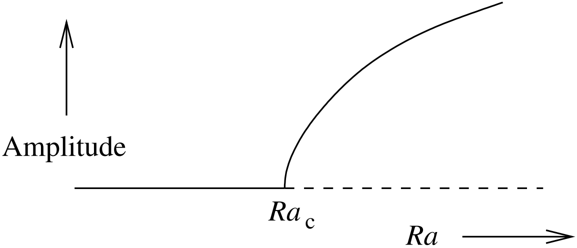
We will not go into the details here. We look for solutions near marginal stability where we expect solutions to be small. We pose expansions

$$\psi = \epsilon\psi_0 + \epsilon^2\psi_1 + \epsilon^3\psi_2 + \cdots$$

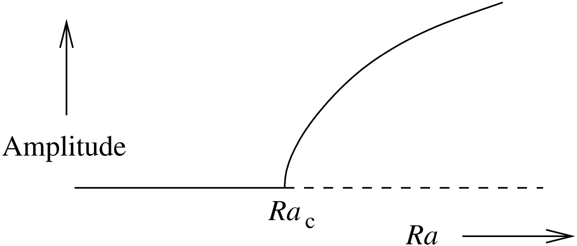
$$T = \epsilon T_0 + \epsilon^2 T_1 + \epsilon^3 T_2 + \cdots$$

$$Ra = \frac{27\pi^4}{4} + \epsilon Ra_1 + \epsilon^2 Ra_2 + \cdots$$

We find $Ra_1 = 0$ and $Ra_2 > 0$



In this analysis you find $\psi_1 = 0$ and $T_1 \propto \sin 2\pi y$.



In this analysis you find $\psi_1 = 0$ and $T_1 \propto \sin 2\pi y$.

Convection tends to make interior of the fluid more evenly mixed, thus reducing the effective temperature gradient and the effective Rayleigh number.

Time dependent problem

You can look for the time variation of these problems to find out how the system evolves in these situations. A simple way of looking at this is assume

$$\psi = a(t) \sin \alpha x \sin \pi y,$$

$$T = b(t) \cos \alpha x \sin \pi y + c(t) \sin 2\pi y.$$

These are substituted into the governing equations. Where the nonlinear terms generate other terms of a different spatial form they are neglected.

This yields a set of equations

$$\frac{da}{dt} = \sigma\left[\frac{\sqrt{2}Ra}{3\pi}b - \frac{3\pi^2}{2}a\right]$$

$$\frac{db}{dt} = \frac{\pi}{\sqrt{2}}a - \frac{3\pi^2}{2}b - \frac{\pi^2}{\sqrt{2}}ac$$

$$\frac{dc}{dt} = \frac{\pi^2}{2\sqrt{2}}ab - 4\pi^2 c$$

where we use $\alpha = \pi/\sqrt{2}$.

With some rescaling of t and the variables this reduces to the Lorenz equations

$$\frac{da}{dt} = \sigma\left[b - a\right]$$

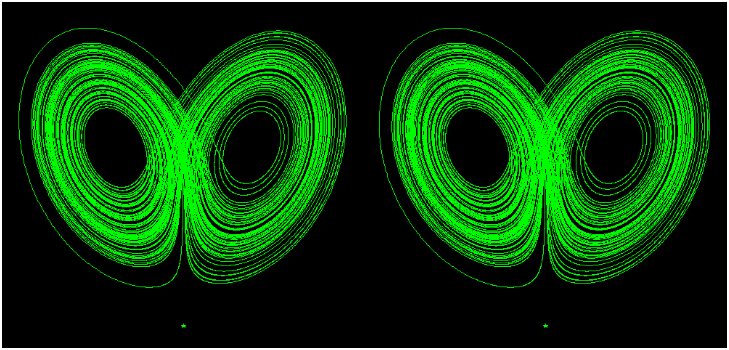
$$\frac{db}{dt} = ra - b - ac$$

$$\frac{dc}{dt} = ab - \frac{8}{3}c$$

where $r = Ra/(27\pi^4/4)$.

These equations are a rational model for the behaviour of convection for small amplitudes, i.e., for r just over 1.

However when you put in large values such as $r = 28$ you get more interesting behaviour....



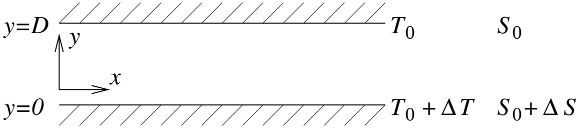
Stereogram — cross your eyes for a 3-D image

$$r = 28$$

Play/Pause

Very pretty, but not a lot to do with thermal convection. But it didn't stop people trying to investigate possible links between the chaos found in the Lorenz attractor and real convection.

The same approach can be used to look at a reduced model for the double-diffusive convection problem where a salinity gradient is also applied across the fluid layer.



Da Costa, Knobloch & Weiss (1981) assumed the same form for the temperature and salinity perturbation, and derived a 5$^{\text{th}}$-order set of equations:

$$\frac{da}{dt} = \sigma\left(-a + r_T b - r_S d\right)$$

$$\frac{db}{dt} = -b + a(1 - c)$$

$$\frac{dc}{dt} = \varpi(-c + ab)$$

$$\frac{dd}{dt} = -\tau d + a(1 - e)$$

$$\frac{de}{dt} = \varpi(-\tau e + ad)$$

With d and e removed these are equivalent to the Lorenz equations previously seen.

Here chaos was found for relatively small amplitude solutions, but again reality got in the way. The initial instability was oscillatory.

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But

$$\cos \alpha x \cos \omega t = \frac{1}{2} \left(\cos(\alpha x - \omega t) + \cos(\alpha x + \omega t) \right)$$

A standing wave is the sum of a right and left travelling wave. Bretherton (1983) showed that if you do the weakly nonlinear analysis to a higher order and allowed the left and right travelling waves to have their separate amplitudes, then one wave would grow and the other decay. No standing oscillatory solution will be observed and instead only travelling waves will be seen (without the rich structure of chaos).

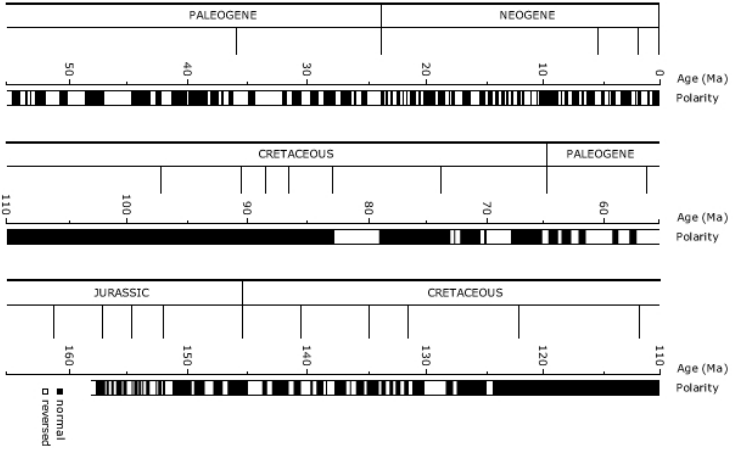
Earth's magnetic field

The previous problem had a sound theoretical background, but was sometimes pushed a bit too far.

Here we will briefly look at an idealized model of a vastly more complicated problem — the generation of the Earth's magnetic field.

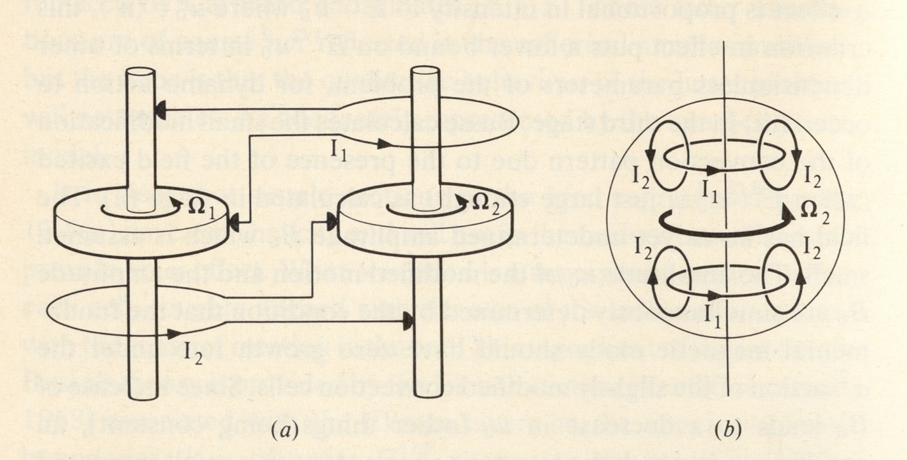
- The Earth's core is too hot for it to be a permanent magnet.
- The convection in the molten metallic core can generate magnetic fields spontaneously...
- ... but it needs rotation etc.
- There has to be a strong toroidal element to the mean field in the core (which we will not see on the surface), and a poloidal part that comes of surface.
  The convection generates the poloidal field from then toroidal field, and the toroidal field from poloidal field.

- 400 times in the last 330,000,000 years the polarity of the Earth's magnetic field has reversed.



From "Fundamentals of Geophysics", by William Lowrie (1997)

Rikitake (1958) proposed a model:

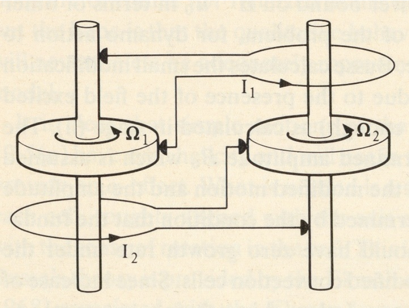


From "Magnetic field generation in electrically conducting fluids", by H.K. Moffatt

Equations for current:

$$L\frac{dI_1}{dt} + RI_1 = M\Omega_1 I_2$$

$$L\frac{dI_2}{dt} + RI_2 = M\Omega_2 I_1$$

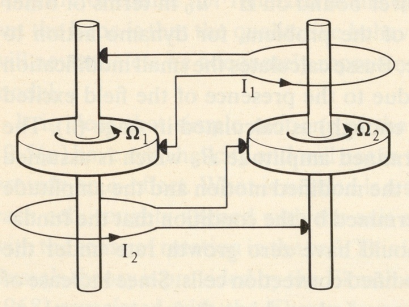


where L is the self-inductance $2\pi M$ is the mutual inductance between the circuits, and R the resistance.

Equations for angular momentum:

$$C\frac{d\Omega_1}{dt} + RI_1 = G - MI_1I_2$$

$$C\frac{d\Omega_2}{dt} + RI_1 = G - MI_1I_2$$



where C is the moment of inertial for each disc.
$-MI_1I_2$ represents the effect of the Lorentz force on the disc, and $M\Omega_1I_2$ the voltage generated by the dynamo effect.

These can be non-dimesionalised to give

$$\frac{dX}{dt} + \mu X = ZY$$

$$\frac{dY}{dt} + \mu Y = ZX$$

$$\frac{dZ}{dt} = \frac{dV}{dt} = 1 - XY$$

Clearly

$$Z - V = \text{const.} = A$$

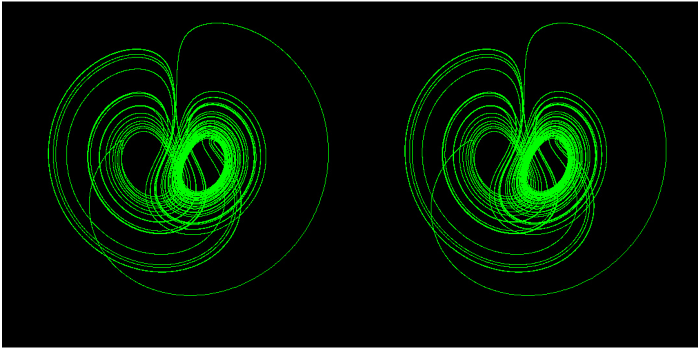
Steady state solutions exist where K is a constant such that

$$A = \mu(K^2 - K^{-2})$$

then

$$X = \pm K, \quad Y = \pm K^{-1}, \quad Z = \mu K^2, \quad V = \mu K^{-2}$$

It can be shown (using a bit more than a linear analysis of the steady solution) that for $K > 1$ there is instability.



$$\mu = 1, \quad K = 2.$$

Stereogram — cross your eyes for a 3-D image

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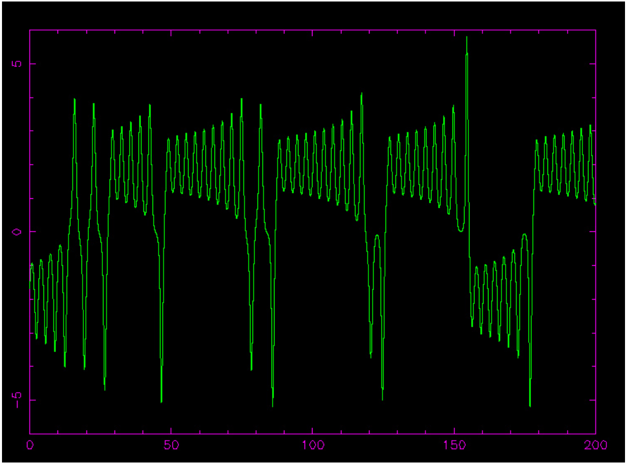


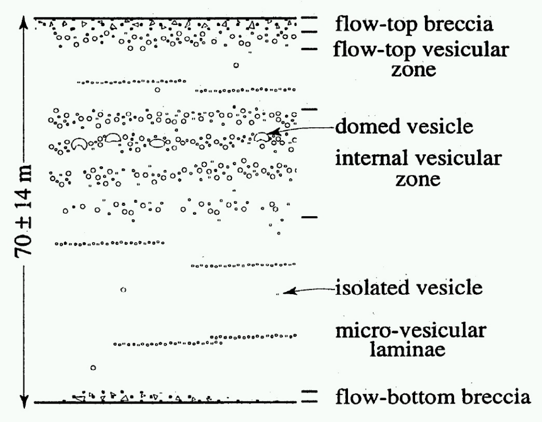
$$\mu = 1, \quad K = 2.$$

This model show the magnetic fields oscillating around ± 2, with occasional flips between these values.

Not a very realistic model, but it does show that spontaneous field reversal in a system that generates its own magnetic field is possible.

From Guinness to Rocks

The last model I will look at shows how very different fields can have a direct influence on each other.



From Manga (1996)

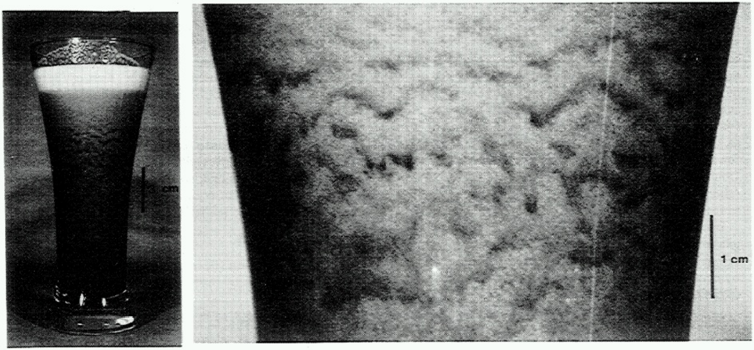


Figure 2. Bubbles in a glass of Guinness beer a few seconds after the beer was poured. The bubbles are initially homogeneously distributed. Horizontal and downward propagating layers of bubbles develop. Secondary Rayleigh-Taylor instabilities form from the layers.

From Manga (1996)

Play/Pause

Flow past a bubble/droplet

Suppose we have two immiscible fluids with densities ρ_{I} and ρ_{O} for the densities of fluids inside and outside. Let their viscosities be μ_{I} and μ_{O} respectively.
(picture)
If the bubble/droplet is small, how fast will it rise/fall?
Small size and speed of bubbles implies $Re \ll 1$, so we have Stokes's flow with

$$0 = -\nabla p + \mu_i \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

As the bubble/droplet is small surface tension will ensure it is almost exactly spherical (what shape is a rain drop?)
We require the tangential stresses to be continuous across the boundary, and no flow across the boundary.
Pressure can be discontinuous across the boundary with adjustments being made by small deviations from the bubble/droplet being spherical.

Following Batchelor (1967): Assume the droplet/bubble instantaneously has its centre at the origin, and velocity $\mathbf{U}$. Because the problem is linear the pressure must be linearly dependent on $\mathbf{U}$, and since the operators in the equations are independent of coordinate system the pressure deviation must be of the form

$$p/\mu_i = (\mathbf{U}\cdot\mathbf{x}) \times \frac{1}{a^2} \times F\left(\frac{\mathbf{x}\cdot\mathbf{x}}{a^2}\right)$$

where a is the radius of the droplet/bubble.
Since $\nabla^2 p = 0$ the pressure will be of the form

$$p/\mu_i = \frac{C_i \mathbf{U}\cdot\mathbf{x}}{r^3} + D_i \mathbf{U}\cdot\mathbf{x}$$

Clearly outside the droplet/bubble D_i will be zero (decays to ∞) and inside C_i will be zero (no singularity at the origin).
We can also use a stream function ψ, and use a spherical polar coordinate system with the axis given by $\theta = 0$ pointing in the direction of $\mathbf{U}$. Then

$$\psi = U \sin^2 \theta \, f(r)$$

Outside the droplet/bubble

$$\frac{d^2 f_{\mathrm{O}}}{dr^2} - \frac{2 f_{\mathrm{O}}}{r^2} = -\frac{C_{\mathrm{O}}}{r}$$

and inside

$$\frac{d^2 f_{\mathrm{I}}}{dr^2} - \frac{2 f_{\mathrm{I}}}{r^2} = \frac{D_{\mathrm{I}} r^2}{2}$$

Boundary conditions

- $f_{\mathrm{O}}/r^2 \to 0$ as $r \to \infty$.
- $\mathbf{U} \cdot \mathbf{x} = 0$ on $r = a$ requires $f_i(a) = a^2/2$.
- No singularity at the origin.

These give

$$f_{\mathrm{O}} = \frac{C_{\mathrm{O}} r}{2} + \left(\frac{a^3}{2} - \frac{C_{\mathrm{O}} a^2}{2} \right) \frac{1}{r}$$

$$f_{\mathrm{I}} = \frac{D_{\mathrm{I}} r^4}{20} + \left(\frac{1}{2} - \frac{D_{\mathrm{I}} a^2}{20} \right) r^2$$

Continuity of stress and velocity give

$$C_{\mathrm{O}} = \frac{a}{2} \frac{2\mu_{\mathrm{O}} + 3\mu_{\mathrm{I}}}{\mu_{\mathrm{O}} + \mu_{\mathrm{I}}}, \qquad D_{\mathrm{I}} = -\frac{5}{a^2} \frac{\mu_{\mathrm{O}}}{\mu_{\mathrm{O}} + \mu_{\mathrm{I}}}.$$

By calculating the resultant force exerted on the droplet/bubble and balancing it with gravity enables one to calculate the speed of fall/rise of the droplet/bubble. This speed is given by

$$V = -\frac{1}{3}\,\frac{a^2\rho_{\mathrm{O}}g}{\mu_{\mathrm{O}}}\left(\frac{\rho_{\mathrm{I}}}{\rho_{\mathrm{O}}} - 1\right)\frac{\mu_{\mathrm{O}} + \mu_{\mathrm{I}}}{\mu_{\mathrm{O}} + \frac{3}{2}\mu_{\mathrm{I}}}$$

Limiting cases

1. Droplet/bubble very low viscosity: $\mu_{\mathrm{I}}/\mu_{\mathrm{O}} \to 0$

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2. Sphere rigid: $\mu_{\rm I}/\mu_{\rm O} \to \infty$

$$V = -\frac{2}{9}\,\frac{a^2\rho_{\rm O} g}{\mu_{\rm O}}\left(\frac{\rho_{\rm I}}{\rho_{\rm O}} - 1\right)$$

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3. Bubble: $\rho_{\rm I}/\rho_{\rm O} \to 0$

$$V = \frac{1}{3}\,\frac{a^2\rho_{\rm O}g}{\mu_{\rm O}}\,\frac{\mu_{\rm O} + \mu_{\rm I}}{\mu_{\rm O} + \frac{3}{2}\mu_{\rm I}}$$

Or combining...

- Rigid bubble ($\mu_{\rm I} \to \infty$, $\rho_{\rm I} \to 0$):

$$V = \frac{2}{9}\frac{a^2 \rho_{\rm O} g}{\mu_{\rm O}}$$

- Inviscid bubble ($\mu_{\rm I} \to 0$, $\rho_{\rm I} \to 0$):

$$V = \frac{1}{3}\frac{a^2 \rho_{\rm O} g}{\mu_{\rm O}}$$

For air rising in water we have $\mu_{\rm O} \approx 10^{-3}\mathrm{kg\,m^{-1}s}$ and $\mu_{\rm I} \approx 1.8 \times 10^{-5}\mathrm{kg\,m^{-1}s}$, So which approximation should we take?

Despite having $\mu_{\mathrm{I}}/\mu_{\mathrm{O}} \ll 1$ it is often appropriate to take the rigid limit!

Despite having $\mu_\mathrm{I}/\mu_\mathrm{O} \ll 1$ it is often appropriate to take the rigid limit!

Surfactants — molecules that accumulate at the interface between the fluids — although low in overall concentration can have a significant impact on the dynamics of the surface, and can effectively stop it flowing.
For Guinness probably rigid spheres, for basaltic magmas probably inviscid bubble.

The above law assumes fluid is at rest at ∞ and sphere is isolated. But as the bubbles go up in Guinness the surrounding fluid must go down. If the bubble velocities are v, and the liquid velocity is u then

$$v - u = \frac{2}{9}\,\frac{a^2\rho_{\mathrm{O}} g}{\mu_{\mathrm{O}}}$$

But this will only be true for dilute bubble suspensions, where bubble-bubble interactions are negligible. However, for the moment we will just assume it holds for general ϕ.

If the proportion of the bubbly liquid occupied by the bubbles is ϕ, then since there is no net up or down flow at any point

$$\phi v + (1-\phi)u = 0$$

or

$$v - u = v + \frac{\pi v}{1-\phi} = \frac{v}{1-\phi}$$

Then the flux of gas upwards is

$$Q(\phi) = \phi v = \frac{2}{9}\,\frac{\phi(1-\phi)a^2\rho_O g}{\mu_O}$$

Conservation of bubble volume gives

$$\frac{\partial \phi}{\partial t} + \frac{\partial Q}{\partial z} = \frac{\partial \phi}{\partial t} + Q'(\phi)\frac{\partial \phi}{\partial z} = 0$$

From this we see ϕ is constant on characteristics given by

$$\frac{dz}{dt} = Q'(\phi)$$

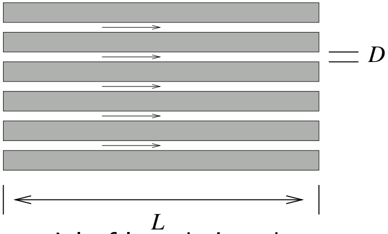
Since Q is non-negative and is zero at $\phi = 0$ and $\phi = 1$ it must have a negative gradient for large enough ϕ, and so the characteristics point down. Waves in the bubble density will travel downwards.

However, for bigger values of ϕ this model is not valid. Flow is more like fluid passing through a porous medium — clearly the case in the head on top.

Porous media — Carman-Kozeny equation

Here we will look at how to model flow through a porous medium. Assume that the flow inside is low Reynolds number Stokes flow.

The Kozeny model



Take a block of material of length L and cross-sectional area A. Assume parallel circular tubes of diameter D, with total area of the passages A'. The fraction of the volume occupied by the fluid is

$$\epsilon = \frac{A'}{A}$$

If the flow through the block is Q, then we can define a flow speed

$$U = Q/A$$

Note: This is not the same as the fluid speed, or even the average fluid speed.

If there is a pressure drop across the block of Δp, then we can find the flow in each tube. If x is the distance along a tube, and r is the distance from its axis then for steady parallel flow we have

$$\frac{\partial p}{\partial x} = \mu \nabla^2 u = \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

With the boundary condition $u = 0$ on $r = D/2$, this gives

$$u = -\frac{\partial p}{\partial x} \frac{\left((D/2)^2 - r^2 \right)}{4\mu}$$

and a flow down each tube of

$$q = -\frac{\partial p}{\partial x} \frac{\pi}{8\mu} \left(\frac{D}{2} \right)^4$$

The number of tubes is

$$\frac{A'}{\pi D^2/4}$$

so the total flow is

$$Q = UA = -\frac{\partial p}{\partial x}\frac{A'D^2}{32\mu}$$

Or the pressure gradient is given by

$$\frac{\partial p}{\partial x} = -\frac{32\mu AU}{A'D^2} = -\frac{32\mu U}{\epsilon D^2}$$

The Carman model

In the average porous medium the passages along which the fluid flows are not parallel tubes. This has a couple of effects

- If L' is the "average length" of the passages then the flow down each tube would be reduced by a factor L/L'.

$$\frac{\partial p}{\partial x} = -\frac{32\mu U L'}{L \epsilon D^2}$$

The total volume is Q, the volume of the tubes is Q', and the volume of the solid is Q_s, then

$$Q' = \epsilon Q, \qquad Q_s = (1 - \epsilon) Q.$$

$$Q' = \frac{\epsilon Q_s}{1 - \epsilon}$$

If we have n tubes, then the surface area of the tubes is

$$S = n\pi D L'$$

Volume of tubes is

$$Q' = \frac{n\pi D^2 L'}{4}$$

So

$$D = \frac{4\epsilon Q_s}{S(1-\epsilon)}$$

Now assume we have m spherical particles of diameter d. Then

$$Q_s = m\frac{\pi d^3}{6}, \qquad S = m\pi d^2$$

So

$$\frac{Q_s}{S} = \frac{d}{6}$$

and

$$D = \frac{2\phi d}{3(1-\epsilon)}$$

Assume that we can use this, and

$$\frac{\partial p}{\partial x} = -\frac{72\mu U L'(1-\epsilon)^2}{L\epsilon^3 d^2}$$

Of course there is no clear definition of L'/L, so people rely on experiments, and it turns out the equation works if $L'/L \approx 2.5$, so

$$\frac{\partial p}{\partial x} = -\frac{180\mu U(1-\epsilon)^2}{\epsilon^3 d^2}$$

(and so L'/L becomes *exactly* $2\frac{1}{2}$!)

Note: Here the fluid volume fraction is ϵ. For comparison with previous results for bubbles and thinking of the fluid filtering through the foam we would have $\epsilon = 1 - \phi$.
Also, for vertical flows with z measuring upwards, the pressure gradient term is replaced by

$$\frac{\partial p}{\partial z} + \rho g$$

With the pressure gradient given by the hydrostatic pressure

$$\frac{\partial p}{\partial z} = -(1-\phi)\rho g$$

and so

$$\phi \rho g = -\frac{180 \mu U \phi^2}{(1-\phi)^3 d^2}$$

The average velocity of the fluid is $U' = U/\epsilon = U/(1-\phi)$, and if the average velocity of the bubbles is V', then

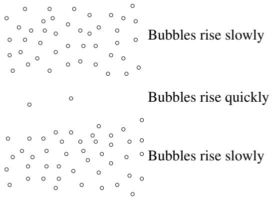
$$\phi V' + (1-\phi)U' = 0$$

So

$$V' = -\frac{(1-\phi)U'}{\phi} = \frac{(1-\phi)^4 d^2 g\rho}{180\mu\phi^3}$$

The $1/180$ factor is significantly smaller than either $1/3$ or $2/9$, indicating that the bubble speed will drop significantly more than our first estimate for higher bubble densities.
This means that we will get downward characteristics for lower ϕ too.

Maybe a hand-waving argument would have been better



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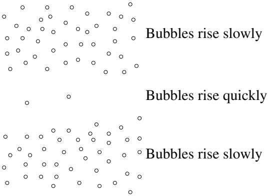

- Bubbles close together rise slowly.

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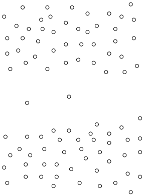

- Bubbles close together rise slowly.
- Bubbles far apart rise quickly.

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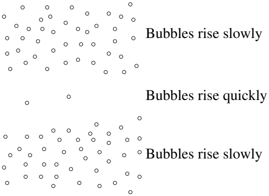

- Bubbles close together rise slowly.
- Bubbles far apart rise quickly.
- Bubbles in gap catch up with bubble layers, and join on the bottom.
- Bubbles on top of the bubble layers can escape.

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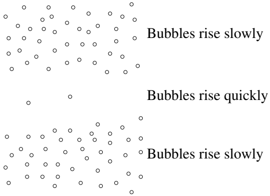
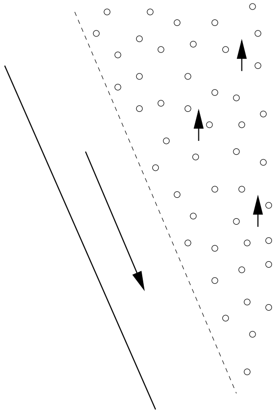
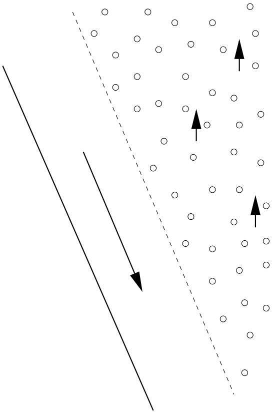


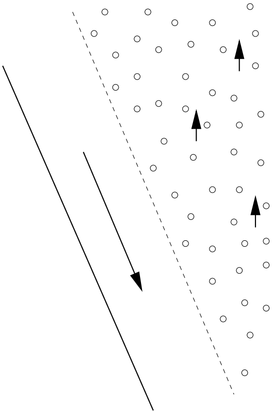
- Bubbles close together rise slowly.
- Bubbles far apart rise quickly.
- Bubbles in gap catch up with bubble layers, and join on the bottom.
- Bubbles on top of the bubble layers can escape.
- Top of layer erodes, and bottom is replenished — layer moves down.

There are other theories...

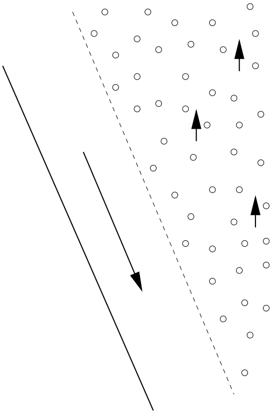




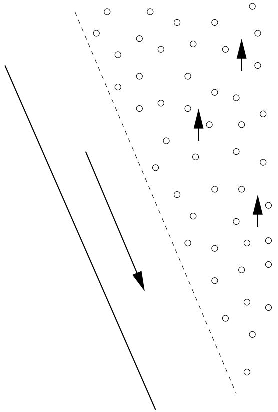
- Glasses have sloping walls.



- Glasses have sloping walls.
- Bubbles rise up, leaving bubble free region near wall.



- Glasses have sloping walls.
- Bubbles rise up, leaving bubble free region near wall.
- Bubble free region is denser and sinks.



- Glasses have sloping walls.
- Bubbles rise up, leaving bubble free region near wall.
- Bubble free region is denser and sinks.
- Instabilities at the edge of the bubble region will look like they are going down.

Which is correct?

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