Mathematical Methods: Complex Variables 1

Answer Sheet

1. (a) $3+7$ (b) $3+2i$

   (c) $5+i$ (d) $8-i$

   (e) -10 (f) $-\frac{7}{5}-\frac{4i}{5}$

2. (a) $\frac{1}{2}$ (b) $-\frac{9}{13}$

   (c) $\frac{2xy}{x^2+y^2}$ (d) $a^n - \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!} - \cdots$

3. $z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) = 0$. Taking real and imaginary parts gives

$$(1) \quad x_1x_2 - y_1y_2 = 0 \qquad \text{and} \qquad (2) \quad x_1y_2 + x_2y_1 = 0.$$

   Assume, without loss of generality, that $z_1 \neq 0$. Then adding $x_1\times(1)$ to $y_1\times(2)$ gives $x_2(x_1^2+y_1^2) = 0$. Since $z_1 \neq 0$ this implies $x_2 = 0$. Similary subtracting $y_1\times(1)$ from $x_1\times(2)$ gives $y_2(x_1^2+y_1^2) = 0$ and so $y_2 = 0$. Hence if $z_1 \neq 0$ then $z_2 = 0$ as required.

4. (a) $\sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{i\pi/4}$ (b) $i = 1e^{i\pi/2}$

   (c) $\frac{24}{5} - \frac{7}{5}i = 5\exp\left(-i\tan^{-1}\left(\frac{7}{24}\right)\right)$ (d) $-3 = 3e^{i\pi}$

   (e) $i = 1e^{i\pi/2}$ (f) $\frac{-i\sin\phi}{1+\cos\phi} = -i\tan\frac{\phi}{2} = \tan\frac{\phi}{2}\,e^{-i\pi/2}$

5. $$r(\cos\theta + i\sin\theta)\times r'(\cos\theta' + i\sin\theta')$$

$$= rr'(cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\sin\theta\cos\theta' + \sin\theta'\cos\theta) = rr'(\cos(\theta+\theta') + i\sin(\theta+\theta')).$$

   Statement in question is obviouslly true for $n = 1$. If true for n then from above result (with $r = r' = 1$)

$$(\cos\theta + i\sin\theta)^{n+1} = (\cos\theta + i\sin\theta)\times(\cos\theta + i\sin\theta)^n$$

$$= (\cos\theta + i\sin\theta)\times(\cos n\theta + i\sin n\theta) = \cos(n+1)\theta + i\sin(n+1)\theta.$$

   Hence it is also true for $n+1$, and by induction true for all n.

$$\cos 5\theta = \mathrm{Re}(\cos 5\theta + i\sin 5\theta) = \mathrm{Re}\left((\cos\theta + i\sin\theta)^5\right)$$

$$= \mathrm{Re}\left(\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\right)$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

6. Can use previous result with $z = \cos\theta + i\sin\theta$ to show $z^5 = 1 \Rightarrow 5\theta = 2n\pi$. Hence $\theta = 0, \pm\frac{2}{5}\pi, \pm\frac{4}{5}\pi, \ldots$. These points give the corners of a pentagon whose corners lie on the unit circle, with one corner at 1.

7.
$$\overline{z_1/z_2} = \overline{\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2})} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} - i(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2})$$
$$= \frac{(x_1-iy_1)(x_2+iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{x_1-iy_1}{x_2-iy_2} = \overline{z_1}/\overline{z_2}.$$
$$|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = z_1\overline{z_1} \pm z_1\overline{z_2} \pm z_2\overline{z_1} + z_2\overline{z_2}.$$

8. Hence
$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2z_1\overline{z_1} + 2z_2\overline{z_2} = 2(|z_1|^2 + |z_2|^2).$$

9. $|z+1| > |z-1| \Rightarrow |z+1|^2 > |z-1|^2 \Rightarrow (z+1)(\overline{z}+1) > (z-1)(\overline{z}-1)$
$\Rightarrow z+\overline{z} > -z-\overline{z} \Rightarrow \mathrm{Re} z > 0$

   Geometrically this inequality is satisfied by all points that are closer in the complex plane to 1 than to -1, hence it gives all points in the right-half plane.

10. $\left|\frac{a-b}{a-\overline{b}}\right| < 1 \Leftrightarrow |a-b|^2 < |a-\overline{b}|^2 \Leftrightarrow a\overline{a} - a\overline{b} - \overline{a}b + b\overline{b} < a\overline{a} - ab - \overline{a}\overline{b} + b\overline{b}$
$\Leftrightarrow ab - a\overline{b} - \overline{a}b + \overline{a}\overline{b} < 0 \Leftrightarrow (a-\overline{a})(b-\overline{b}) < 0 \Leftrightarrow \mathrm{Im}a \times \mathrm{Im}b > 0.$

11. Recall $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2i$. Substitute in and rearrange to get $z(a/2 - ib/2) + \overline{z}(a/2+ib/2) = c$, which is of the form given when $B = a/2 + ib/2$.

12. The circle is given by the formula $|z-a| = r$ or $(z-a)\overline{(z-a)} = r^2$. This can be rearranged to give $z\overline{z} - \overline{a}z - a\overline{z} + a^2 - r^2 = 0$, which is of the required form if $B = -a$ and $C = a^2 - r^2$. The circle passes through the origin when $C = 0$.

13. Let $w = 1/z$ then $z = 1/w$. Substitute into the equation for a line: $B/\overline{w} + \overline{B}/w = c$. If $c \neq 0$ this can be rearranged to $w\overline{w} - (B/c)w - (\overline{B}/c)\overline{w} = 0$, which is the equation of a circle passing through the origin. If $c = 0$ then you get $wB + \overline{w}\overline{B} = 0$, the equation of another line passing through the origin.

    Similarly for the circle you get after substitution $w\overline{w} + (B/C)w + (\overline{B}/c)\overline{w} + 1/C = 0$ provided $C \neq 0$. Again this is the equation of a circle. If $C = 0$ you get $Bw + \overline{B}\overline{w} + 1 = 0$, the equation of a line that doesn't pass through the origin.

14. (a) $\dfrac{x(x-1)+y^2}{(x-1)^2y^2}, \quad \dfrac{-y}{(x-1)^2+y^2}, \quad 1, \quad -1$

    (b) $x^2 - y^2 - 3y - 3, \quad 2xy + 3x, \quad -6, \quad 5$

    (c) $x^4 - 6x^2y^2 + y^4, \quad 4x^3y - 4xy^3, \quad -4, \quad 0$

    (d) $\dfrac{(x+1)^2 - y^2}{(x+1)^2+y^2}, \quad \dfrac{-2(x+1)y}{(x+1)^2+y^2}, \quad 3/25, \quad -4/25$

15. (a) If $f(z) = \overline{z}$ then $f(z+\Delta z) - f(z) = \overline{z+\Delta z} - \overline{z} = \overline{\Delta z}$. Hence, given any $\epsilon > 0$ we can choose δ equal to ϵ. Then $\forall |\Delta z| < \delta$, $|f(z+\Delta z) - f(z)| < \epsilon$. I.e. $f(z)$ is continuous. Note the choice of δ here is independent of z, and so $\overline{z}$ is uniformly continuous.

    (b) $|f(z+\Delta z) - f(z)| = ||z+\Delta z| - |z||$. If $|z+\Delta z| > |z|$, then, using the triangle inequality, $|f(z+\Delta z) - f(z)| = |z+\Delta z| - |z| \leq |z| + |\Delta z| - |z| = |\Delta z|$. If $|z+\Delta z| < |z|$, then $|f(z+\Delta z) - f(z)| = |z| - |z+\Delta z| = |z| - |z - (-\Delta z)|$. From the triangle inequality we can show that $|a-b| \geq |a| - |b|$, and so $|f(z+\Delta z) - f(z)| = |z| - |z-(-\Delta z)| \leq |z| - |z| + |\Delta z| = |\Delta z|$. So as in example (a) If we choose δ equal to ϵ then $\forall |\Delta z| < \delta$, $|f(z+\Delta z) - f(z)| < \epsilon$.

    (c) If $f(z) = z^2$ then $f(z+\Delta z) - f(z) = 2z\Delta z + \Delta z^2$. If $\delta \leq 1$ then $|\Delta z| < \delta$ implies $|2z\Delta z + \Delta z^2| \leq |2z\Delta z| + |\Delta z^2| < (2|z| + \delta)\delta \leq (2|z|+1)\delta$. Hence, given any $\epsilon > 0$ if we choose δ to be the smaller of 1 or $\epsilon/(1+2|z|)$ then we will have $|f(z+\Delta z) - f(z)| < \epsilon$ for all $|\Delta z| < \delta$. So $f(z) = z^2$ is continuous for all values of z, including the ones in the region $|z| < 1$. To show uniform continuity we note that for $|z| < 1$ $\epsilon/(1+2|z|) > \epsilon/3$ and so if we set $\delta = \epsilon/3$ then $|f(z+\Delta z) - f(z)| < \epsilon$ for all values of z in the disc if $|\Delta z| < \delta$, hence $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

16. If anyone managed to evaluate either (b) or (c) correctly without recourse to complex variable techniques I would like to see their solutions. The answers were (a) $2\pi/\sqrt{3}$, (b) $\pi/3$ and (c) $\frac{1}{2}\sqrt{\pi/2}$. You will be given a handout later that will show how to evaluate these integrals using complex variable methods. To do the integral (a) use the standard substitution $t = \tan\theta/2$, then $\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$ and $\tan\theta = 2t/(1-t^2)$. So

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \int_{-\pi}^{\pi} \frac{d\theta}{2+\sin\theta} = \int_{-\infty}^{\infty} \frac{1}{2+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt = \int_{-\infty}^{\infty} \frac{2}{2+2t^2+2t}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{2}{3/2+(2t+1)^2/2}\,dt = \frac{4}{3}\int_{-\infty}^{\infty} \frac{1}{1+((2t+1)/\sqrt{3})^2}\,dt$$

$$= \frac{4}{3}\left[\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_{-\infty}^{\infty} = \frac{4}{3} \times \frac{\sqrt{3}}{2} \times \pi = \frac{2\pi}{\sqrt{3}}.$$