Mathematical Methods: Complex Variables

A function has at most one power series expansion

To prove this we first show that **a power series is continuous at the origin**. I.e. we want to show that

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \sum_{n=0}^{\infty} a_n z^n = f(0) = a_0.$$

If $R > 0$ is the radius of convergence of the power series then the power series converges absolutely for $0 < r < R$, and so

$$\sum_{n=1}^{\infty} |a_n| r^{n-1} = \frac{1}{r} \sum_{n=1}^{\infty} |a_n| r^n$$

converges. Then, for all z such that $0 < |z| < r$

$$|f(z) - a_0| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \leq \sum_{n=1}^{\infty} |a_n z^n| = |z| \sum_{n=1}^{\infty} |a_n||z|^{n-1} \leq |z| \sum_{n=1}^{\infty} |a_n| r^{n-1}.$$

But this last summation converges to some constant $S > 0$, and so given any $\epsilon > 0$ we can choose a δ such that (i) $\delta < r$ and (ii) $\delta < \epsilon/S$. Then

$$|f(z) - a_0| < \epsilon, \quad \forall |z| < \delta.$$

This shows that $f(z)$ is continuous at $z = 0$

Next we show that **a power series is unique**. Suppose that $f(z)$ has two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n.$$

From the above result $f(z)$ is continuous at the origin, and in particular

$$f(0) = a_0 = b_0.$$

So we have shown the first two elements in the power series are the same. We use induction to show that $a_n = b_n$ for all n. Assume $a_n = b_n$ for all n to $n = k$, then we can cancel all the terms up to $a_k z^k$ to give

$$\sum_{n=k+1}^{\infty} a_n z^n = \sum_{n=k+1}^{\infty} b_n z^n.$$

For $z \neq 0$ we can divide both sides by z^{k+1} to give

$$a_{k+1} + a_{k+2} z + a_{k+3} z^2 + \cdots = b_{k+1} + b_{k+2} z + b_{k+3} z^2 + \cdots.$$

We can again appeal to continuity at the origin to show that $a_{k+1} = b_{k+1}$. Hence by induction $a_n = b_n$ for all n, and so the power series representation is unique.

Term-by-term differentiation of a power series

Here we show that if you differentiate a power series term-by-term then the resulting power series has the same radius of convergence as the original power series.

If you differentiate the power series $\sum a_n z^n$ term-by-term you obtain the series

$$\sum_{n=1}^{\infty} n a_n z^{n-1}.$$

If R is the radius of convergence of the initial power series then for any z_0 such that $0 < |z_0| < R$ the original series converges, and so $|a_n z_0^n| \to 0$ as $n \to \infty$. In particular we can find an N such that $|a_n| < |z_0|^{-n}$ for all $n > N$. Thus if M is an integer greater that N

$$\sum_{n=M}^{\infty} \left| n a_n z^{n-1} \right| = \sum_{n=M}^{\infty} n |a_n| |z|^{n-1} \leq \sum_{n=M}^{\infty} n \frac{|z|^{n-1}}{|z_0|^n}.$$

This last summation converges by the ratio test for all $|z| < |z_0|$, and so the power series obtained by term-by-term differentiation also converges. We can choose $|z_0|$ as close to R as we wish, hence we have absolute convergence for all $|z| < R$.

If $|z| > R$ then $a_n z^n \not\to 0$ and so $n a_n z^n \not\to 0$ also. Thus R is also the radius of convergence for the differentiated series.

This argument can be repeated to give the result that if you differentiate the power series term-by-term as many times as you like the resulting power series will always have the same radius of convergence as the original series.

A similar argument shows that if you integrate the power series term-by-term the resulting power series will also have the same radius of convergence.

A power series is an analytic function

In this section we will show that if a function $f(z)$ defined by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has a nonzero radius of convergence then it is analytic within its radius of convergence, and its derivative is given by

$$f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

To prove this consider

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \left(\frac{(z+\Delta z)^n - z^n}{\Delta z} - n z^{n-1} \right).$$

But

$$\frac{(z+\Delta z)^n - z^n}{\Delta z} - n z^{n-1} = \Delta z \left[(z+\Delta z)^{n-2} + 2z(z+\Delta z)^{n-3} + \cdots + (n-1) z^{n-2} \right]. \tag{1}$$

This last result is not obvious. We prove it by induction:

When $n = 2$ we see that (1) is true as

$$\frac{(z+\Delta z)^2 - z^2}{\Delta z} - 2z = \Delta z$$

as required.

Suppose that (1) is true for $n = k$ then

$$\begin{aligned}
&\frac{(z+\Delta z)^{k+1} - z^{k+1}}{\Delta z} - (k+1)z^k \\
&= \frac{(z+\Delta z)^{k+1} - (z+\Delta z)z^k + (z+\Delta z)z^k - z^{k+1}}{\Delta z} - (k+1)z^k \\
&= \frac{(z+\Delta z)\left((z+\Delta z)^k - z^k\right)}{\Delta z} + \frac{(z+\Delta z - z)z^k}{\Delta z} - (k+1)z^k \\
&= (z+\Delta z)\frac{(z+\Delta z)^k - z^k}{\Delta z} + z^k - (k+1)z^k \qquad \text{[Substitute result for } n = k\text{]} \\
&= (z+\Delta z)\left\{kz^{k-1} + \Delta z\left[(z+\Delta z)^{k-2} + 2z(z+\Delta z)^{k-3} + \cdots + (k-1)z^{k-2}\right]\right\} - kz^k \\
&= \Delta z\left[(z+\Delta z)^{k-1} + 2z(z+\Delta z)^{k-2} + \cdots + (k-1)(z+\Delta z)z^{k-2}\right] - kz^k + kz^{k-1}(z+\Delta z) \\
&= \Delta z\left[(z+\Delta z)^{(k+1)-2} + 2z(z+\Delta z)^{(k+1)-3} + \cdots + ((k+1)-2)z^{(k+1)-3}\right] + \Delta z k z^{k-1} \\
&= \Delta z\left[(z+\Delta z)^{(k+1)-2} + 2z(z+\Delta z)^{(k+1)-3} + \cdots + ((k+1)-2)z^{(k+1)-3} + ((k+1)-1)z^{(k+1)-2}\right]
\end{aligned}$$

and so (1) is true for $n = k+1$. By induction it is true for all n.

If we consider

$$\sum_{n=2}^{\infty} a_n \Delta z\left[(z+\Delta z)^{n-2} + 2z(z+\Delta z)^{n-3} + \cdots + (n-1)z^{n-2}\right]$$

then the brackets contain $n-1$ terms whose largest coefficient is $n-1$. We can also find some R_1 such that both $|z| \le R_1 < R$ and $|z + \Delta z| \le R_1 < R$ for some sufficiently small Δz. In this case the absolute value of this series is bounded by

$$\sum_{n=2}^{\infty} |a_n|(n-1)^2 R_1^{n-2} \le \sum_{n=2}^{\infty} |a_n| n(n-1) R_1^{n-2}.$$

But we know that the power series for $f(z)$ differentiated term by term converges absolutely for all points inside the circle of convergence. Since $R_1 < R$ the series on the right must converge to give a sum, $S(R_1)$ say. Hence

$$\left|\frac{f(z+\Delta z) - f(z)}{\Delta z} - f_1(z)\right| \le |\Delta z| S(R_1).$$

Letting $\Delta z \to 0$ we note that since R_1 is arbitrary this implies that $f(z)$ is analytic at all points in the interior of the circle of convergence, and that its derivative is given by the series with term-wise differentiation.

We have now shown that all power series represent analytic functions, all we need to do now is show that all analytic functions can be represented by power series ...