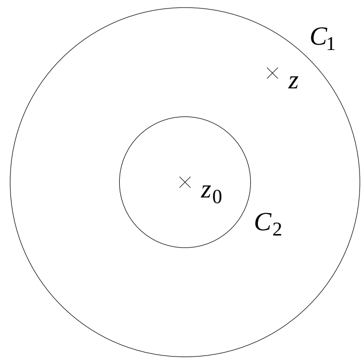
Mathematical Methods: Complex Variables

Laurent Series

If $f(z)$ is analytic on the two concentric circles C_1 and C_2 with centre z_0 an in the region inbetween them, then $f(z)$ can be represented by the Laurent Series



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}}\, dz' \qquad \text{and} \qquad b_n = \frac{1}{2\pi i} \oint_C (z' - z_0)^{n-1} f(z')\, dz'$$

where C is a contour circuiting z_0 in the annulus between C_1 and C_2.

Proof

If z lies inside the annulus then from Cauchy's Integral theorem

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z}\, dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z' - z}\, dz'.$$

Since z lies inside C_1 then, using the same argument as for the Taylor series,

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z}\, dz' = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}}\, dz'.$$

Note: As $f(z)$ may not be analytic inside C_2 we *cannot* use Cauchy's integral formula.

To find the second set of terms we proceed in a similar manner to the derivation in the Taylor series, but this time we note that on C_2

$$|z' - z_0| < |z - z_0|.$$

Hence

$$\frac{1}{z' - z} = \frac{1}{(z' - z_0) - (z - z_0)} = -\frac{1}{(z - z_0)\left(1 - \dfrac{z' - z_0}{z - z_0}\right)}$$

$$= -\frac{1}{z - z_0}\left(1 + \frac{z' - z_0}{z - z_0} + \left(\frac{z' - z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{z' - z_0}{z - z_0}\right)^n\right) - \frac{1}{z - z'}\left(\frac{z' - z_0}{z - z_0}\right)^{n+1}$$

Substituting into the integral and integrating gives

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')}{z'-z}\,dz' = \frac{1}{2\pi i}\left(\frac{1}{z-z_0}\oint_{C_2} f(z')dz' + \frac{1}{(z-z_0)^2}\oint_{C_2}(z'-z_0)f(z')dz' + \cdots \right.$$
$$\left. + \frac{1}{(z-z_0)^{n+1}}\oint_{C_2}(z'-z_0)^n f(z')dz'\right) + R_n(z).$$

As before we can show that $R_n(z) \to 0$ as $n \to \infty$ and so

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')}{z'-z}\,dz' = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

where

$$b_n = \frac{1}{2\pi i}\oint_C (z'-z_0)^{n-1} f(z')\,dz'.$$

The integrals for a_n and b_n can both be deformed to the common contour C to obtain the original expression.

To show that the Laurent series converges in the annulus found by expanding C_1 until it reaches a singularity and decreasing C_2 until it reaches a singularity we split

$$f(z) = g(z) + h(z)$$

where

$$g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$
$$h(z) = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

Then $g(z)$ is analytic inside C_1 and $h(z)$ is analytic outside C_2 [Why?]. Hence the expansion for $g(z)$ is valid up to the first singularity outside C_1, where the expansion for $h(z)$ is also valid, and so we can extend the region of validity of the expansion to the first singularity in $f(z)$ which will also be the first singularity in $g(z)$.

Similarly we can shrink C_2 until it encounters a singularity.