Mathematical Methods: Complex Variables

Recall the question:

Especially for those that think they are very good at integration. Evaluate, without using any complex variable theory,

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}, \qquad (a) \qquad\qquad \int_0^{\infty} \frac{dx}{x^6+1}, \qquad (b)$$

$$\int_0^{\infty} \sin x^2\, dx. \qquad (c)$$

Here is how to do these integrals using contour integration.

(a) Let $z = e^{i\theta}$ then $dz = ie^{i\theta}d\theta$ or $d\theta = -iz^{-1}dz$. Recall $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - z^{-1}\right)$. Then θ ranging from 0 to 2π corresponds to z moving around the circle of radius 1 centred on the origin. We will denote this contour as C. Hence

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_C \frac{-iz^{-1}dz}{2+\frac{1}{2i}\left(z-z^{-1}\right)} = \oint_C \frac{2\,dz}{z^2+4iz-1}$$

$$= -\oint \frac{i\,dz}{\sqrt{3}(z+2i-\sqrt{3}i)} + \oint \frac{i\,dz}{\sqrt{3}(z+2i+\sqrt{3}i)} = 2\pi i \times \frac{-i}{\sqrt{3}} + 2\pi i \times 0 = \frac{2\pi}{\sqrt{3}}.$$

(b) Let C be the closed contour going along the real axis from 0 to R, then in a semicircular arc in the upper half of the complex plane from R to $-R$, and then along the real axis from $-R$ to 0. Denote each of these sections C_1, C_2 and C_3 respectively. Then

$$\oint_C \frac{dz}{z^6+1} = \oint_{C_1} \frac{dz}{z^6+1} + \oint_{C_2} \frac{dz}{z^6+1} + \oint_{C_3} \frac{dz}{z^6+1}.$$

For $R > 1$ the integral around C consists of the contributions from the poles at $z = e^{\frac{\pi}{6}i}$, i and $e^{\frac{5\pi}{6}i}$. These give

$$\oint_C \frac{dz}{z^6+1} = 2\pi i \times \left(\frac{1}{6}e^{\frac{5\pi i}{6}} + \frac{-i}{6} + \frac{1}{6}e^{-\frac{\pi i}{6}}\right) = \frac{\pi i}{3}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2} - i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{2\pi}{3}$$

But as $R \to \infty$ the contribution from C_2 tends to 0, while the contributions from both sections C_1 and C_3 tend towards the integral that we are seeking. Hence

$$\int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}.$$

(c) In this case we find both the integral given and $\int_0^\infty \cos x^2\, dx$ at the same time. We note that $e^{iz^2} = \cos z^2 + i \sin z^2$ and consider the contour of integration, C, made up of three parts; C_1 along the real axis from 0 to R, C_2 the arc of the circle of radius R centred on 0 going from R to $Re^{i\pi/4}$, and C_3 the straight line from $Re^{i\pi/4}$ to 0. Note that e^{iz^2} is analytic for *all* z, and so the integral around this contour is 0.

If we look at the limit of these three integrals as $R \to \infty$ we see that

$$\int_{C_1} e^{iz^2} dz \to \int_0^\infty \cos x^2 \, dx + i \int_0^\infty \sin x^2 \, dx$$

and

$$\int_{C_3} e^{iz^2} dz \to \int_\infty^0 e^{-t^2} e^{i\pi/4} dt = -\frac{1+i}{\sqrt{2}} \int_0^\infty e^{-t^2} \, dt = -\frac{(1+i)}{\sqrt{2}} \frac{\sqrt{\pi}}{2}.$$

We have used here the substitution $z = e^{i\pi/4} t$.

Lastly we show that the contribution from the last part of the contour, C_2, decays to 0 as $R \to \infty$.

$$\int_{C_2} e^{iz^2} dz = \int_0^{\pi/4} \exp\left(iR^2 e^{2i\theta}\right) iRe^{i\theta} \, d\theta = \int_0^{\pi/4} \exp\left(iR^2 \left(\cos 2\theta + i \sin 2\theta\right)\right) iRe^{i\theta} \, d\theta$$

$$= \int_0^{\pi/4} \exp\left(iR^2 \cos 2\theta + i\theta\right) \exp\left(-R^2 \sin 2\theta\right) iR \, d\theta.$$

Looking at the modulus of this

$$\left| \int_{C_2} e^{iz^2} dz \right| \le \int_0^{\pi/4} \left| \exp\left(iR^2 \cos 2\theta + i\theta\right) \exp\left(-R^2 \sin 2\theta\right) iR \right| \, d\theta$$

$$= \int_0^{\pi/4} \exp\left(-R^2 \sin 2\theta\right) R \, d\theta.$$

But $\sin 2\theta \ge 4\theta/\pi$ for $0 \le \theta \le \pi/4$, and so

$$\left| \int_{C_2} e^{iz^2} dz \right| \le \int_0^{\pi/4} \exp\left(-R^2 \sin 2\theta\right) R \, d\theta$$

$$\le \int_0^{\pi/4} \exp\left(-4R^2\theta/\pi\right) R \, d\theta = \frac{\pi}{4R}\left(1 - e^{-R^2}\right).$$

Hence we see that as $R \to \infty$ the contribution from this contour vanishes. This gives

$$\oint_C e^{iz^2} \, dz = \int_0^\infty \cos x^2 \, dx + i \int_0^\infty \sin x^2 \, dx - \frac{(1+i)}{2}\sqrt{\frac{\pi}{2}} = 0.$$

Finding real and imaginary parts gives the required result

$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$