Mathematical Methods: Complex Variables

Summation of infinite series

Here we will look at how we can use contour integration to sum infinite series.

The first thing we do is to consider

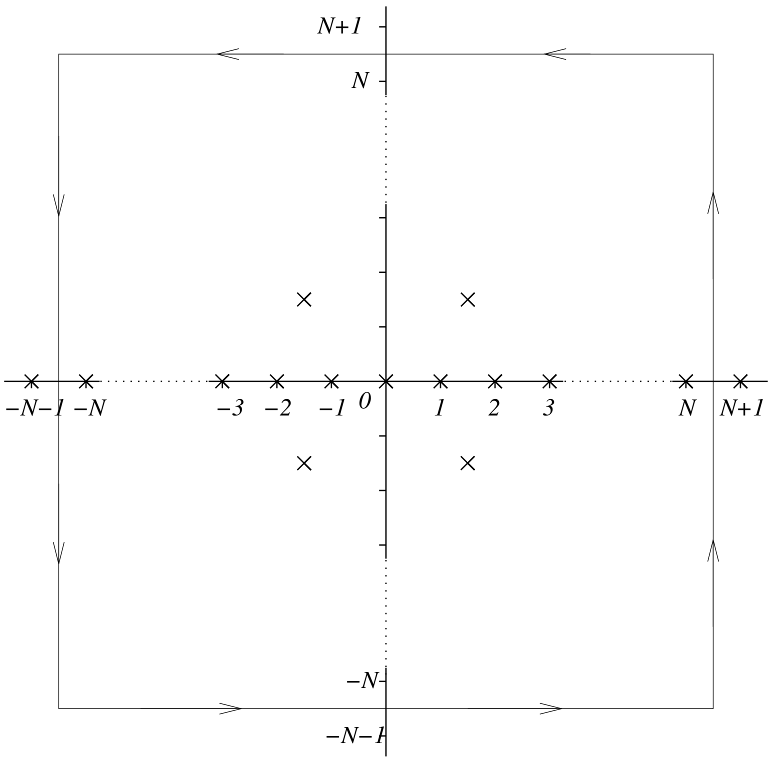
$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z}$$

This has singularities when $\sin \pi z = 0$, i.e., when $z = 0, \pm 1, \pm 2, \pm 3, \ldots$. The residue at each of these points is

$$\operatorname*{Res}_{z=n} \frac{\cos \pi z}{\sin \pi z} = \left. \frac{\cos \pi z}{\pi \cos \pi z} \right|_{z=n} = \frac{1}{\pi}$$

If we have an analytic function $f(z)$, then $f(z) \cot \pi z$ would have residues of $f(n)/\pi$ at each of these points provided $f(z)$ was not singular at $z = n$. If we could do an integration around all these points then we may be able to pick contributions of $f(n)/\pi$ from each of these singularities on the real axis. We cannot do a circular contour and let is radius increase as we have done before because the contour would keep passing through singularities. Instead we will consider a contour which increases in size in discrete steps.

Consider the following square contour, C:



On the vertical lines of this square z will have real parts $\pm(N + 1/2)$, as do the imaginary parts of z on the horizontal lines. If we integrate around this contour we will clearly pick up all the residues from the integers from $-N$ to N, and also all the residues of the poles, $\times$, of $f(z)$ (provided N is large enough for C to include them all). If we can show that as $N \to \infty$

$$\oint_C f(z) \cot \pi z \, dz \to 0$$

then the sum of all these residues will also be 0.

First we will examine $\cot \pi z$ on this contour. We will consider the contour in two different parts. If $z = x + iy$, the first part will be for $|y| > 1/2$.

$$|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$$

$$= \left| \frac{e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}}{e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}} \right| = \left| \frac{e^{i\pi x - \pi y} + e^{-i\pi x + \pi y}}{e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}} \right|$$

Next we use

$$|a + b| \leq |a| + |b|$$

and so

$$|\cot \pi z| \leq \frac{|e^{i\pi x - \pi y}| + |e^{-i\pi x + \pi y}|}{|e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}|}.$$

To deal with the terms on the bottom will use

$$|a| - |b| \leq |a - b|$$

From this we can deduce that if $|a| > |b|$ then

$$\frac{1}{|a - b|} \leq \frac{1}{|a| - |b|}.$$

and if $|b| > |a|$ then

$$\frac{1}{|a - b|} \leq \frac{1}{|b| - |a|}.$$

These can be combined to give

$$\frac{1}{|a - b|} \leq \frac{1}{||a| - |b||}$$

provided $|a| \neq |b|$, hence

$$|\cot \pi z| \leq \frac{|e^{i\pi x - \pi y}| + |e^{-i\pi x + \pi y}|}{||e^{i\pi x - \pi y}| - |e^{-i\pi x + \pi y}||}$$

But $|e^{x+iy}| = e^x$ so

$$|\cot \pi z| \leq \frac{e^{-\pi y} + e^{\pi y}}{|e^{-\pi y} - e^{\pi y}|}$$

For $y > 1/2$

$$|\cot \pi z| \leq \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}}$$

Similarly, for $y < -1/2$

$$|\cot \pi z| \leq \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}}$$

So the magnitude of $\cot \pi z$ is bounded away from the region around the real axis.

Now we look at the region $-1/2 \leq y \leq 1/2$. We have to take into account the real part of z here since we know it is in this region that we have singularities here. We consider the case $z = N + 1/2 + iy$ where N is an integer,

$$|\cot \pi z| = \left| \frac{e^{i\pi x - \pi y} + e^{-i\pi x + \pi y}}{e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}} \right| = \left| \frac{e^{i\pi(N+1/2) - \pi y} + e^{-i\pi(N+1/2) + \pi y}}{e^{i\pi(N+1/2) - \pi y} - e^{-i\pi(N+1/2) + \pi y}} \right|$$

$$= \left| \frac{e^{i\pi(2N+1) - \pi y} + e^{\pi y}}{e^{i\pi(2N+1) - \pi y} - e^{\pi y}} \right| = \left| \frac{e^{i\pi(2N+1)} e^{-\pi y} + e^{\pi y}}{e^{i\pi(2N+1)} e^{-\pi y} - e^{\pi y}} \right|$$

But $e^{i\pi(2N+1)} = -1$, so

$$|\cot \pi z| = \left| \frac{-e^{-\pi y} + e^{\pi y}}{-e^{-\pi y} - e^{\pi y}} \right| = \left| \frac{e^{\pi y} - e^{-\pi y}}{e^{\pi y} + e^{-\pi y}} \right| = |\tanh \pi y| < 1$$

Combining these results gives

$$|\cot \pi z| < \frac{1 + e^{-\pi}}{1 - e^{-\pi}}$$

for all points on the contour.

Returning to our integral, provided $f(z)$ decays fast enough, all we require is that we can find M and $k > 1$ such that $|f(z)| < M/R^k$ where $R = |z| > R_0$. The length of the contour is $8N + 4$

$$\left| \oint_C f(z) \cot \pi z \, dz \right| \leq \text{Length} \times \text{maximum of } |f(z) \cot \pi z| \text{ on } C$$

$$\left| \oint_C f(z) \cot \pi z \, dz \right| \leq (8N + 4) \times \frac{M}{(N + 1/2)^k} \to 0 \quad \text{as} \quad N \to \infty.$$

And so, taking the limit $N \to \infty$ we find

$$\oint_C f(z) \cot \pi z \, dz = 2\pi i \left(\sum_{n=-\infty}^{\infty} \operatorname*{Res}_{z=n} f(z) \cot \pi z + \sum_j \operatorname*{Res}_{z=z_j} f(z) \cot \pi z \right) = 0$$

where the second summation is over all the singularities of $f(z)$, excluding those that are on integer values as these have already been counted in the first summation.

Example: Find

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Let $f(z) = 1/z^2$. Then we have a pole of order 3 at $z = 0$. The contributions from $z = -n$ and $z = n$ are the same, so

$$2 \sum_{n=1}^{\infty} \operatorname*{Res}_{z=n} \frac{\cot \pi z}{z^2} + \operatorname*{Res}_{z=0} \frac{\cot \pi z}{z^2} = 0,$$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}+\operatorname*{Res}_{z=0}\frac{\cot\pi z}{z^2}=0.$$

But

$$\cot\pi z=\frac{1}{\pi z}-\frac{\pi z}{3}-\frac{\pi^3 z^3}{45}-\frac{2\pi^5 z^5}{945}-\cdots$$

so the residue of $\cot\pi z/z^2$ is $-\pi/3$. Hence

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

We can also see, using $f(z)=1/z^4$ and $f(z)=1/z^6$ respectively, that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\qquad\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$$