Mathematical Methods: Complex Variables 3

1. Which of the following are harmonic, and hence could be the real part of an analytic function:

   (a) $xy + x + y$ (b) $\dfrac{y}{(x+1)^2 + y^2}$

   (c) $(y\cos x - x\sin x)\, e^{-y}$ (d) $(y\cos x + x\sin x)\, e^{-y}$

   For the three functions that could be the real part of an analytic function find the corresponding conjugate harmonic function $v(x, y)$, and hence find $f(z)$ (i.e. find an expression for f that only depends on the variable z, and not on x, y or $\overline{z}$.)

2. Calculate explicitly each of the integrals

   (a) $\displaystyle\int_C \overline{z}\, dz$ (b) $\displaystyle\int_C z^2 + 2iz\, dz$

   for each of the two curves from 0 to $2 + 2i$ given by (i) the straight line from 0 to 2 followed by the straight line from 2 to $2+2i$, and (ii) the parabola $z(t) = x + iy$ where x and y are given by $x = 2t$ and $y = 2t^2$ for $0 \leq t \leq 1$. Verify your answers for (b) by finding the indefinite integral of $z^2 + 2iz$.

3. [Question 8 from 1990 exam] Under what conditions does the formula

   $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \quad n = 1, 2, \ldots$$

   give the derivatives of $f(z)$ at $z = z_0$?

   Prove that if $|f(z)| \leq M$ on the circle $|z - z_0| = r$ then

   $$\left| f^{(n)}(z_0) \right| \leq \frac{n! M}{r^n}.$$

   If $f(z) = e^z$ find the corresponding value of M on the circle $|z| = 1$ and use the inequality to predict an upper bound for $|f^{(n)}(0)|$. Comment on its effectiveness.

   Show that for any entire function $f(z)$ for which $|f(z)|$ is bounded for all z then $f(z)$ must be a constant [Liouville's Theorem].

   The function of the real variable, $f(x) = \cos x$ is analytic everywhere and bounded for all x, but is not constant; explain why this does not contradict Liouville's Theorem.

4. State and prove Cauchy's Integral Formula for the value of a function at points inside a contour in terms of its values on the contour. State the formula for

the higher derivatives of the function at points inside the contour. Use this to calculate

(a) $\displaystyle\int_C \frac{\cosh 2z}{z^2+1}dz$ (b) $\displaystyle\int_C \frac{\sin 2z}{z^2(z^2+1)}dz$

where C is the circle of radius 2 centred on $z=0$, traversed in an anticlockwise direction.

5. Derive Gauss' Mean Value Theorem.

6. State and prove the Maximum Modulus Theorem.

   [Hint: The point with maximum modulus must be either on the boundary or in the interior of the region under consideration. Consider a small circle around an interior point, and use Gauss' Theorem to show it is not an isolated maximum. Note also that Kreyszig and Spiegel have errors in their proofs.]

7. Prove the Minimum Modulus Theorem. [Hint: Consider $1/f(z)$]

8. Two lines γ_1 and γ_2 intersect at the point z_0 in the z-plane with angle ψ. Show that if $f(z)$ is an analytic function then the two lines in the w-plane given by $w=f(z_i)$ for all z_i that lie on the lines γ_i also intersect with angle ψ.

9. (a) State what is meant by saying that the function $f(z)$, where $z=x+iy$ and x,y are real, is differentiable at a point z.

   (b) If $f(z)$ is differentiable at z prove the Cauchy-Riemann equations

   $$u_x = v_y, \qquad u_y = -v_x,$$

   where $f(z)=u(x,y)+iv(x,y)$.

   (c) by considering limits as $z+\delta z$ approaches z radially and also along $|z+\delta z|=r$ show that if

   $$f(z)=u(r,\theta)+iv(r,\theta)$$

   is differentiable at $z=re^{i\theta}$ $(r\neq 0)$ then

   $$ru_r = v_\theta, \qquad rv_r = -u_\theta.$$

   If $v(r,\theta)=r^{-2}\cos 2\theta$ find $u(r,\theta)$ and hence deduce the general form for $f(z)$ by solving these equations.