Mathematical Methods: Complex Variables 1
Answer Sheet, 2006–7

1. (a) $\nabla^2(x^2+y^2) = 4$, not harmonic
   (b) $\nabla^2(y^2-x^2) = 0$, harmonic. $u_x = -2x = v_y$, so $v = -2xy + g(x)$. $u_y = 2y = -u_x = 2y - g'(x)$, so $g'(x) = 0$, $g(x) = C$, and $v(x,y) = -2xy + C$. $f(z) = -z^2 + iC$
   (c) $\nabla^2(\sin 2x \sinh y) = -3\sin 2x \sinh y$, not harmonic.
   (d) $\nabla^2(\cosh x \cos y) = 0$, harmonic. $u_x = \sinh x \cos y = v_y$, so $v = \sinh x \sin y + g(x)$. $u_y = -\cos x \sin y = -v_x = -\cos x \sin y - g'(x)$, so $g'(x) = 0$, $g(x) = C$, and $v(x,y) = \sinh x \sin y + C$. $f(z) = \cosh z + C$.
   (e) $\nabla^2(x^3y - xy^3) = 0$, harmonic. $u_x = 3x^2y - y^3 = v_y$, so $v = \frac{3}{2}x^2y^2 - y^4/4 + g(x)$. $u_y = x^3 - 3xy^2 = -v_x = -3xy^2 - g'(x)$, so $g'(x) = -x^3$, $g(x) = -x^4/4$, $v = -x^4/x + \frac{3}{2}x^2y^2 - y^4/4 + C$. $f(z) = -iz^4/4 + iC$
   (f) $\nabla^2(\tan^{-1}(x/y) = 0$, harmonic. $u_x = y/(x^2+y^2) = v_y$ so $v = \frac{1}{2}\ln(x^2+y^2) + g(x)$. $u_y = -x/(x^2+y^2) = -v_x = -x/(x^2+y^2) + g'(x)$, so $g'(x) = 0$, $g(x) = C$, and $v(x,y) = \frac{1}{2}\ln(x^2+y^2) + C$. $f(z) = \tan^{-1}(\infty) + i\ln z + iC = \pi/2 + i\ln z + iC$.
   (iv) Here
$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}, \qquad \text{and} \qquad v = -\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$
   Hence
$$u_x = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = v_y, \quad \text{and} \quad u_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = -v_x.$$
   And so $f(z)$ is analytic. $[f(z) = \cot z]$

2. $|f(z)|^2 = u(x,y)^2 + v(x,y)^2 = C$, where C is a constant. Taking partial derivatives with respect to x and y gives
$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0, \qquad \text{and} \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$
   Use the Cauchy-Riemann equations to eliminate the partial derivatives of v:
$$uu_x - vu_y = 0, \qquad uu_y + vu_x = 0.$$
   Then add u times the first equation to v times the second equation to eliminate u_y, giving
$$(u^2+v^2)u_x = 0.$$
   If $u^2 + v^2 = 0$ then $u = v = 0$ and so $f(z) = 0$ and is constant. Otherwise $u_x = v_y = 0$. Similarly $u_y = -v_x = 0$, and so u and v are both constant, hence $f(z)$ is a constant.

3. (a) $(n+2)/(3n+2i) \to 1/3$ as $n \to \infty$, since the individual terms do not tend to zero the series cannot converge.

   (b) By the ratio test, $|a_{n+1}/a_n| = |\frac{(n+1)(1+i)}{2n}| \to \left|\frac{1+i}{2}\right| = 1/\sqrt{2} < 1$ so the series converges.

   (c) By the ratio test $a_{n+1}/a_n = \frac{(n+1)^{2(n+1)}}{(n+1)!} \times \frac{n!}{n^{2n}} = \frac{(n+1)^{2n+1}}{n^{2n}} = (\frac{n+1}{n})^{2n}(n+1)$. But $\frac{n+1}{n} > 1$ and so $a_{n+1}/a_n > n+1 > 1$ for all n. Hence the series diverges.

   (d) Both the root test and the ratio test fail to say whether this series diverges or converges. Probably the easiest way to see that this series converges is to consider the real and imaginary parts separately. In both cases you get a sequence of numbers converging to zero with alternating signs, and hence each subseries converges and so the whole series converges.

4. (a) There are two cases:

   Taylor series for $|z| < 1$:

$$\frac{2}{1-z^2} = 2\left(1 + z^2 + z^4 + z^6 + \cdots\right) = 2 + 2z^2 + 2z^4 + 2z^6 + \cdots.$$

   Laurent series for $|z| > 1$:

$$\frac{2}{1-z^2} = -\frac{2}{z^2(1-(1/z^2))} = -\frac{2}{z^2}\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right) = -\frac{2}{z^2} - \frac{2}{z^4} - \frac{2}{z^6} - \frac{2}{z^8} - \cdots$$

   (b) $z^3 + 6z^2 + 11z + 6 = (z+1)(z^2+5z+6) = (z+1)(z+2)(z+3)$. Singularities at $z = -1$, -2 and -3. Hence one Taylor series and 3 Laurent series. Use

$$\frac{1}{z^3 + 6z^2 + 11z + 6} = \frac{1/2}{z+1} - \frac{1}{z+2} + \frac{1/2}{z+3}$$

   Each term has a Taylor and Laurent series

   - $1/(z+1)$
     Taylor: $|z| < 1$ $1/(z+1) = 1 - z + z^2 - z^3 + \cdots$
     Laurent: $|z| > 1$ $1/(z+1) = 1/z - 1/z^2 + 1/z^3 + \cdots$
   - $1/(z+2)$
     Taylor: $|z| < 2$ $1/(z+2) = 1/2 - z/4 + z^2/8 - z^3/16 + \cdots$
     Laurent: $|z| > 2$ $1/(z+2) = 1/z - 2/z^2 + 4/z^3 + \cdots$
   - $1/(z+2)$
     Taylor: $|z| < 3$ $1/(z+3) = 1/3 - z/9 + z^2/27 - z^3/81 + \cdots$
     Laurent: $|z| > 3$ $1/(z+3) = 1/z - 3/z^2 + 9/z^3 + \cdots$

   Add together appropriately to get series

   $|z| < 1$: $1/(z^3 + 6z^2 + 11z + 6) = 1/6 - 11z/36 + 85z^2/216 + \cdots$

   $1 < |z| < 2$: $1/(z^3 + 6z^2 + 11z + 6) = \frac{1}{2}(1/z - 1/z^2 + 1/z^3 + \cdots) - (1/2 - z/4 + z^2/8 - z^3/16 + \cdots) + \frac{1}{2}(1/3 - z/9 + z^2/27 - z^3/81 + \cdots)$

   $2 < |z| < 3$: $1/(z^3 + 6z^2 + 11z + 6) = \frac{1}{2}(1/z - 1/z^2 + 1/z^3 + \cdots) - (1/z - 2/z^2 + 4/z^3 + \cdots) + \frac{1}{2}(1/3 - z/9 + z^2/27 - z^3/81 + \cdots)$

   $|z| > 3$: $1/(z^3 + 6z^2 + 11z + 6) = 1/z^3 - 6/z^4 + \cdots$

(c) Laurent series only. Do some simple manipulations first:

$$\frac{e^{-z}}{z-1} = \frac{e^{-(z-1)-1}}{z-1} = e^{-1} \times \frac{e^{-(z-1)}}{z-1} = \frac{e^{-1}}{z-1}\left[1-(z-1)+\frac{(z-1)^2}{2!}-\frac{(z-1)^3}{3!}+\cdots\right]$$

$$= \frac{e^{-1}}{(z-1)} - e^{-1} + \frac{e^{-1}}{2!}(z-1) - \frac{e^{-1}}{3!}(z-1)^2 + \cdots$$

(d) Singularities at $z = -1$ and $z = 1/2 \pm \sqrt{3}/2$. Two Laurent series only.
$|z+1| < \sqrt{3}$: $1/(z^3+1) = 1/(3(z+1)) + 1/3 + 2(z+1)/9 + (z+1)^2/9 + \cdots$
$|z+1| > \sqrt{3}$: $1/(z^3+1) = 1/(z+1)^3 + 3/(z+1)^4 + 6/(z+1)^5 + \cdots$