Mathematical Methods: Complex Variables

Coursework 2 Answer Sheet, 2007–08

1. (a) Poles at $z = 1$ and $z = -1$ (order 1).

$$\operatorname*{Res}_{z=-1} \frac{z^2+1}{z^2-1} = -1, \qquad \operatorname*{Res}_{z=1} \frac{z^2+1}{z^2-1} = 1$$

(b) Pole at $z = 1$ (order 5)

$$\operatorname*{Res}_{z=1} \frac{\sin z}{(z-1)^5} = \lim_{z\to 1} \frac{1}{4!}\frac{d^4}{dz^4}(z-1)^5 \times \frac{\sin z}{(z-1)^5} = \lim_{z\to 1} \frac{1}{4!}\frac{d^4}{dz^4} \sin z = \lim_{z\to 1} \frac{1}{4!} \sin z = \frac{\sin 1}{4!}$$

(c) Pole at $z = 0$ (order 2) and at $z = \pi/2 + n\pi$ (order 1).

$$\operatorname*{Res}_{z=0} \frac{tanz}{z^2} = \lim_{z\to 0} \frac{1}{2!}\frac{d}{dz} z^2 \times \frac{\tan z}{z^2} = \lim_{z\to 0} \frac{1}{2!}\frac{d}{dz} \tan z = \lim_{z\to 0} \frac{1}{2!} \sec^2 z = \frac{1}{2}$$

$$\operatorname*{Res}_{z=\pi/2+n\pi} \frac{\tan z}{z^2} = \operatorname*{Res}_{z=\pi/2+n\pi} \frac{(\sin z)/z^2}{\cos z} = \lim_{z\to\pi/2+n\pi} \frac{(\sin z)/z^2}{-\sin z} = -\frac{1}{(\pi/2+n\pi)^2}$$

2. (a) Using $z = e^{i\theta}$, with contour C unit circle centred on origin.

$$\int_0^{2\pi} \frac{\sin\theta}{5+4\sin\theta}\, d\theta = \int_C \frac{z^2-1}{2iz(z+2i)(2z+i)} dz$$

Only sigularities at $z = 0$ and $z = -i/2$ inside C, so

$$\int_0^{2\pi} \frac{\sin\theta}{5+4\sin\theta}\, d\theta = 2\pi i \operatorname*{Res}_{z=-i/2} \frac{z^2-1}{4iz(z+2i)(z+i/2)} + 2\pi i \operatorname*{Res}_{z=0} \frac{z^2-1}{4iz(z+2i)(z+i/2)}$$

$$= 2\pi i \frac{-5}{12i} + 2\pi i \frac{1}{4i} = -\frac{\pi}{3}$$

(b) $z^4 + 10z^2 + 9 = (z^2+9)(z^2+1)$. Poles at $z = \pm i$, and $z = \pm 3i$, only $z = i$ and $z = 3i$ lie inside C.

$$\int_{-\infty}^{\infty} \frac{1+2x^2}{x^4+10x^2+9}\, dx = \int_C \frac{1+2z^2}{z^4+10z^2+9}\, dz$$

$$= 2\pi i \operatorname*{Res}_{z=i} \frac{1+2z^2}{z^4+10z^2+9} + 2\pi i \operatorname*{Res}_{z=3i} \frac{1+2z^2}{z^4+10z^2+9}$$

$$= 2\pi i \lim_{z\to i} \frac{1+2z^2}{4z^3+20z} + 2\pi i \lim_{z\to 3i} \frac{1+2z^2}{4z^3+20z}$$

$$= 2\pi i \left[\frac{-1}{-4i+20i} + \frac{-17}{-108i+60i}\right] = 2\pi i \left[\frac{-1}{16i} + \frac{-17}{-48i}\right] = 2\pi i \frac{14}{48i} = \frac{7\pi}{12}.$$

3. Consider $f(z)\cot\pi z$ integrated around the standard square contour contour. This has poles of residues $f(n)/\pi$ at $z = n$, and poles at $z = \pm i$. To get the residues at these last 2 singulaties just evaluate $(\cot\pi z)/2z$ at these points. This gives

$$0 = \sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+1)} + (\cot\pi z)/2z|_{z=i} + (\cot\pi z)/2z|_{z=-i}$$

So

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+1)} = -(\cot \pi i)/2i - (\cot -\pi z)/(-2i)$$

But $\cot \pi i = \cos \pi i / \sin \pi i = \cosh \pi / i \sinh \pi = -i \coth \pi$ and $\cot -\pi i = \cos -\pi i / \sin -\pi i = \cosh \pi / (-i \sinh \pi) = i \coth \pi$, so

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+1)} = -(\coth \pi)/2 - (\coth \pi)/(-2) = \coth \pi$$

But sum on left is not one in question, so we rearrange

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2+1)} = \frac{1}{\pi} + 2\sum_{n=0}^{\infty} \frac{1}{\pi(n^2+1)} = \coth \pi$$

or

$$\sum_{n=0}^{\infty} \frac{1}{\pi(n^2+1)} = \frac{\pi \coth \pi - 1}{2}$$

4.

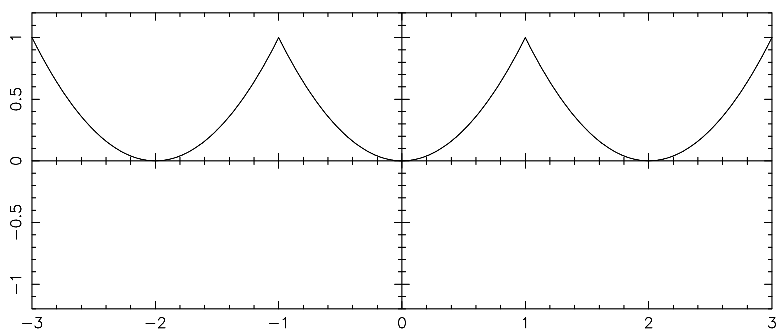
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{p} + b_n \sin \frac{2n\pi x}{p},$$

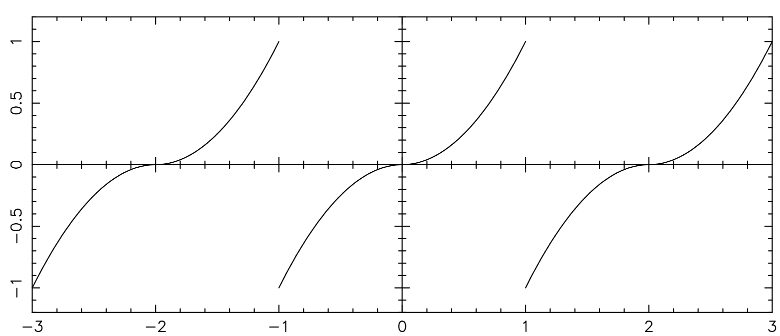
$$a_0 = \frac{1}{p} \int_0^p f(x)\, dx,$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{2n\pi x}{p}\, dx,$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{2n\pi x}{p}\, dx.$$

Graphs with the even function at the top and the odd function at the bottom:





For an even function $b_1 = b_2 = b_2 = \ldots = 0$ and for an odd function $a_0 = a_1 = a_2 \ldots = 0$. Since we are doing the odd case $a_0 = a_1 = a_2 \ldots = 0$. Note: $f(x) \neq x^2$ in the interval $[-1, 1]$, only in the interval $[0, 1]$.

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx = 2 \int_{0}^{1} f(x) \sin n\pi x \, dx = 2 \int_{0}^{1} x^2 \sin n\pi x \, dx$$

$$= 2 \left[-\frac{x^2}{n\pi} \cos n\pi x \right]_0^1 + \frac{2}{n\pi} \int_{-1}^{1} 2x \cos n\pi x \, dx$$

$$= -\frac{2(-1)^n}{n\pi} + \frac{4}{n^2\pi^2} [x \sin n\pi x]_0^1 - \frac{4}{n^2\pi^2} \int_0^1 \sin n\pi x \, dx$$

$$= -\frac{2(-1)^n}{n\pi} + \frac{4}{n^3\pi^3} [\cos n\pi x]_0^1 = -\frac{2(-1)^n}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3}.$$

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{2(-1)^n}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3} \right) \sin n\pi x$$

$$= \left(\frac{2}{\pi} - \frac{8}{\pi^3} \right) \sin \pi x - \frac{2}{2\pi} \sin 2\pi x + \left(\frac{2}{3\pi} - \frac{8}{27\pi^3} \right) \sin 3\pi x - \frac{2}{4\pi} \sin 4\pi x + \cdots.$$