1. (a) Show that

$$\nabla\left(r^n\right) = nr^{n-2}\underline{r},$$

where $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$. Hence find the scalar function $\phi(r)$ such that

$$\nabla\phi = \frac{\underline{r}}{r^3} \quad \text{and} \quad \phi(1) = 0.$$

(b) Show that

$$\underline{F} = \left(y^2\cos x + z^3\right)\underline{i} + (2y\sin x - 4)\,\underline{j} + \left(3xz^2 + 2\right)\underline{k}$$

is a conservative force field. Find the scalar potential for $\underline{F}$ and hence find the work done in moving an object from $(0, 1, -1)$ to $(\pi/2, -1, 2)$ in the presence of the field.

2. (a) The surfaces $x + 2y - z = 10$ and $x^2 + y^2 - z^2 = 9$ meet in a curve. Find the equations of the tangent line and the normal plane to this curve at the point $(2, 3, -2)$.

(b) If $\phi = x+y+xyz$, $\underline{A} = xy^2\underline{i}+yz^2\underline{j}+zx^2\underline{k}$, find at the point $P(1, 2, -3)$ the quantities $\underline{A}\cdot\nabla\phi$, $\phi\nabla\cdot\underline{A}$, $\nabla\phi\times\underline{A}$ and $(\nabla\phi)\times(\nabla\times\underline{A})$.

3. (a) Sketch the area A bounded by the curves $y = x^2$ and $y = -x^2$ and lying between the lines $x = -1$ and $x = 1$ in the x,y plane. Evaluate the double integral

$$\iint_A \left(x^4 - 2y\right)\,dy\,dx.$$

(b) Use triple integration with Cartesian coordinates to find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$.

4. (a) A plane P meets the x, y and z axes at points $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,3)$. If $\underline{F} = xy\underline{i} + x^2\underline{j} + yz\underline{k}$ evaluate

$$\iint_S \underline{F} \cdot \underline{n} \, dS$$

where S is the surface of the triangle ABC and $\underline{n}$ is the upward pointing unit normal vector to S.

(b) If $\underline{A} = 2x\underline{i} + 2y\underline{j} + x^2y^2z^2\underline{k}$ and S is the upper half of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{27} = 1,$$

evaluate

$$\iint_S (\nabla \times \underline{A}) \cdot \underline{n} \, dS$$

either directly or by using Stokes' theorem, where $\underline{n}$ is the unit normal vector drawn out of the ellipsoid.

5. If $f(z) = u(x,y) + iv(x,y)$ is an analytic function state the Cauchy-Riemann equations satisfied by $u(x,y)$ and $v(x,y)$ where $z = x + iy$. Show that u satisfies Laplace's equation, $\nabla^2 u = 0$.

Identify which of the following could be the real part of an analytic function:

(i) $u(x,y) = e^x \cosh y + x^3y - xy^3$,

(ii) $u(x,y) = e^{-x} \sin y - x^3y + xy^3$.

Find the corresponding conjugate harmonic function $v(x,y)$, and hence find the analytic function $f(z) = u(x,y) + iv(x,y)$ in terms of the complex variable z only.

Turn over ...

6. A function $f(x)$ is periodic with period p. Write down its general Fourier expansion along with integral expressions for the coefficients a_0, a_n and b_n with $n \geq 1$.

   The periodic function $f(x)$ has period 4. It is defined in the interval $0 \leq x \leq 2$ by
$$f(x) = x(2 - x).$$

   Sketch this function on the interval $-4 \leq x \leq 6$ for the two cases (i) $f(x)$ is an odd function, and (ii) $f(x)$ is an even function. What can be deduced about the Fourier coefficients in these two cases?

   Given that $f(x)$ is an *even* periodic function, by using integration by parts when required, calculate the Fourier coefficients for the function $f(x)$. Hence write down its Fourier expansion.

   Deduce from this expansion that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7. Use contour integration techniques to evaluate the following integrals, stating clearly the contour used:

   (i)
$$\int_0^{2\pi} \frac{\cos\theta}{5 - 4\cos\theta}\, d\theta,$$

   (ii)
$$\int_{-\infty}^{\infty} \frac{x^2 + 4}{(x^2 + 1)(x^2 + 9)}\, dx.$$

Turn over ...

8. (a) State the ratio test for the convergence of a power series.

By using the ratio test or otherwise, find the radius of convergence, R, for the following power series

(i) $1 + z + \dfrac{1}{2}z^2 + \cdots + \dfrac{1}{n}z^n + \cdots,$

(ii) $1 + z + 2z^2 + \dfrac{9}{2}z^3 + \cdots + \dfrac{n^n}{n!}z^n + \cdots,$

(iii) $1 + z + 2z^2 + 6z^3 + \cdots + n!z^n + \cdots.$

You may quote the result that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

(b) Identify the poles of

$$f(z) = \frac{z^2}{(z^2+1)(z^2-4)}.$$

This function has three different Taylor/Laurent series centred on $z = 0$ with different regions of validity. From the locations of the poles or otherwise identify what these three regions will be.

Find the Taylor/Laurent series for $f(z)$ in each of the three regions, showing terms involving z^n for $-4 \le n \le 4$.

Internal Examiners:	Professor J. Mathon
	Dr O.S. Kerr
External Examiners:	Professor D. J. Needham
	Professor M. E. O'Neill