X2 Mathematical Methods – 2005

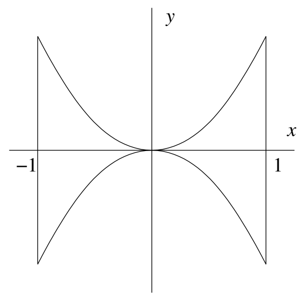
1. (a) $\phi = 1 - 1/r$.

   (b) $\nabla \times \underline{F} = \underline{0}$, so $\underline{F}$ is conservative. $\underline{F} = \nabla\phi = \nabla(y^2 \sin x + xz^3 - 4y + 2z + C)$, work done $= \phi(\pi/2, -1, 2) - \phi(0, 1, -1) = 15 + 4\pi$.

2. (a) Normals (not unit) on surface given by $\underline{N}_1 = (1, 2, -1)$ and $\underline{N}_2 = (2x, 2y, -2z)$. At point $\underline{N}_2 = (4, 6, 4)$. Tangent vector $\underline{T} = \underline{N}_1 \times \underline{N}_2 = (14, -8, -2)$. Tangent line $\underline{x} = (2, 3, -2) + \lambda\underline{T}$, normal plane by $(\underline{x} - (2, 3, -2)) \cdot \underline{T} = 14x - 8y - 2z - 8 = 0$.

   (b) At P, $\phi = -3$, $\underline{A} = (4, 18, -3)$, $\nabla\phi = (1 + yz, 1 + xz, xy) = (-5, -2, 2)$, $\nabla \cdot \underline{A} = y^2 + z^2 + x^2 = 14$, $\nabla \times \underline{A} = (-2yz, -2xz, -2xy) = (12, 6, -4)$. So $\underline{A} \cdot \nabla\phi = -62$, $\phi\nabla \cdot \underline{A} = -42$, $\nabla\phi \times \underline{A} = (-30, -7, -82)$ and $(\nabla\phi) \times (\nabla \times \underline{A}) = (-4, 4, -6)$.

3. (a) Region of integration inside lines:



$$\iint_A \left(x^4 - 2y\right) dy\, dx = \int_{-1}^{1} \int_{-x^2}^{x^2} \left(x^4 - 2y\right) dy\, dx = \int_{-1}^{1} 2x^6\, dx = 4/7.$$

   (b) Intersection of surfaces at $x^2 + 4y^2 = 4$. $V = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_{x^2+3y^2}^{4-y^2} 1\, dz\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} 4 - x^2 - 4y^2\, dy\, dx = \int_{-2}^{2} 8(1-x^2/4)^{1/2} - 2x^2(1-x^2/4)^{1/2} - 8(1-x^2/4)^{3/2}/3\, dx$. Use the substitution $x = 2\sin\theta$ to get $V = \int_{-\pi/2}^{\pi/2} 16\cos^2\theta - 16\sin^2\theta\cos^2\theta - \frac{16}{3}\cos^4\theta\, d\theta = 16\pi/2 - 16\pi/8 - \frac{16}{3} \times 3\pi/8 = 4\pi$

4. (a) $\iint_S \underline{F} \cdot \underline{n}\, dS = 5/4$, (b) $\iint_S (\nabla \times \underline{A}) \cdot \underline{n}\, dS = 0$.

5. $u_x = v_y$, $u_y = -v_x$. $\nabla^2 u = u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$.

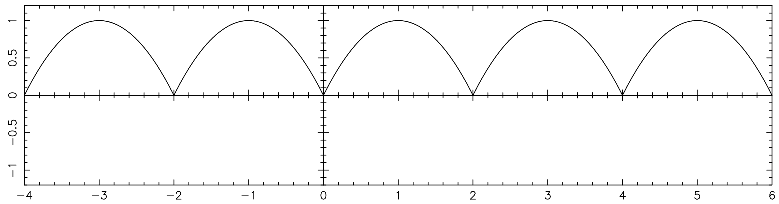
   (i) $\nabla^2 u(x, y) = 2e^x \cosh y$, (ii) $\nabla^2 u(x, y) = 0$, so use (ii). $v(x, y) = e^{-x}\cos y + x^4/4 - 3x^2y^2/2 + y^4/4 + C$, $f(z) = ie^{-iz} + iz^4/x + iC$

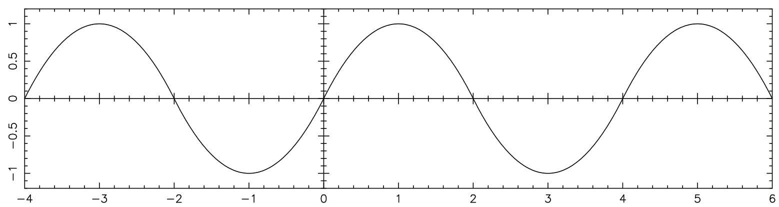
6.

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos\frac{2n\pi x}{p} + b_n \sin\frac{2n\pi x}{p}, \quad a_0 = \frac{1}{p}\int_0^p f(x)\, dx,$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{2n\pi x}{p}\,dx, \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{2n\pi x}{p}\,dx.$$

Graphs with the even function at the top and the odd function at the bottom:





For an even function $b_1 = b_2 = b_2 = \ldots = 0$ and for an odd function $a_0 = a_1 = a_2 = \ldots = 0$.

$a_0 = 2/3$, $a_n = -\dfrac{8(1+(-1)^n)}{n^2\pi^2}$, $f(x) = 2/3 - \frac{4}{\pi^2}\cos\pi x - \frac{4}{2^2\pi^2}\cos 2\pi x - \frac{4}{3^2\pi^2}\cos 3\pi x - \cdots$. Use $f(0) = 0$ to get required result.

7. (i) $\pi/3$, (ii) $7\pi/12$.

8. (a) For $\sum_0^\infty a_n z^n$, if $\lim_{n\to\infty} a_{n+1}/a_n = L$, then series converges absolutely for $|z| < R = 1/L$ and diverges for $|z| > R = 1/L$.

   (i) $R = 1$, (ii) $R = 1/e$, (iii) $R = 0$.

   (b) Poles at $\pm i$ and ± 2. Taylor series for $|z| < 1$, two Laurent series for $1 < |z| < 2$ and $2 < |z|$.

$$f(z) = \frac{z^2}{(z^2+1)(z^2-4)} = \frac{1/5}{z^2+1} + \frac{4/5}{z^2-4}.$$

   For $|z| < 1$, $1/(z^2+1) = 1 - z^2 + z^4 - \cdots$, and for $|z| > 1$ $1/(z^2+1) = z^{-2} - z^{-4} + z^{-6} - \cdots$.

   For $|z| < 2$, $1/(z^2-4) = -1/4 - z^2/16 - z^4/64 - \cdots$, and for $|z| > 2$, $1/(z^2-4) = z^{-2} + 4z^{-4} + 16z^{-6} + \cdots$

   Hence, for $|z| < 1$, $f(z) = -z^2/4 + 3z^4/16 + \cdots$, for $1 < |z| < 2$, $f(z) = \cdots - z^{-4}/5 + z^{-2}/5 - 1/5 - z^2/20 - z^4/80 - \cdots$, and for $|z| > 2$ $f(z) = z^{-2} + 3z^{-4} + \cdots$.