X2 Mathematical Methods – 2007

1. ?????

2. ?????

3. ?????

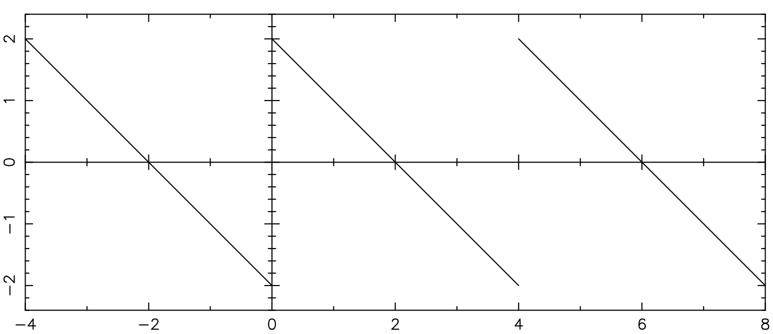
4. ?????

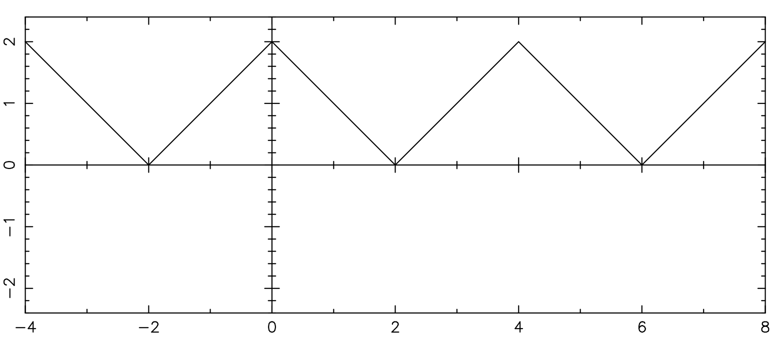
5. $u_x = v_y$, $u_y = -v_x$. $\nabla^2 u = u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$.

   (i) $\nabla^2 u(x,y) = 2e^x \cosh y$, (ii) $\nabla^2 u(x,y) = 0$, so use (ii). $v(x,y) = e^{-x}\cos y + x^4/4 - 3x^2y^2/2 + y^4/4 + C$, $f(z) = ie^{-iz} + iz^4/4 + iC$

6.
$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos\frac{2n\pi x}{p} + b_n \sin\frac{2n\pi x}{p}, \quad a_0 = \frac{1}{p}\int_0^p f(x)\,dx,$$
$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{2n\pi x}{p}\,dx, \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{2n\pi x}{p}\,dx.$$

   Graphs with the odd function at the top and the even function at the bottom:





   For an even function $b_1 = b_2 = b_2 = \ldots = 0$ and for an odd function $a_0 = a_1 = a_2 = \ldots = 0$.

   Since odd $a_0 = a_1 = a_2 = \ldots = 0$.

$$b_n = \frac{2}{4}\int_0^4 (2-x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[(2-x)\left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2}\right]_0^4 - \int_0^4 \frac{1}{n\pi}\cos\frac{n\pi x}{2}\,dx = \frac{4}{n\pi}$$

$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi x}{2} = \frac{4}{\pi} \sin \frac{\pi}{2} + \frac{4}{2\pi} \sin \frac{2\pi}{2} + \frac{4}{3\pi} \sin \frac{3\pi}{2} + \cdots$. Use $f(1) = 1$ to get required result.

7. (i) Using $z = e^{i\theta}$, with contour C unit circle centred on origin.

$$\int_0^{2\pi} \frac{\sin\theta}{5 - 3\sin\theta}\, d\theta = \int_C -\frac{z^2 - 1}{3iz(z - 3i)(z - i/3)} dz$$

Only sigularities at $z = 0$ and $z = i/3$ inside C, so

$$\int_0^{2\pi} \frac{\sin\theta}{5 - 3\sin\theta}\, d\theta = 2\pi i \operatorname*{Res}_{z=i/3} -\frac{z^2 - 1}{3iz(z - 3i)(z - i/3)} + 2\pi i \operatorname*{Res}_{z=0} -\frac{z^2 - 1}{3iz(z - 3i)(z - i/3)}$$

$$= -2\pi i \frac{1}{3i} - 2\pi i \frac{-1/9 - 1}{i/3 - 3i} = -\frac{2\pi}{3} + \frac{5\pi}{6} = \frac{\pi}{6}$$

(ii) Poles at $z = \pm i$, and $z = -1 \pm i$, only $z = i$ and $z = -1 + i$ lie inside C.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 2x + 2)}\, dx = \int_C \frac{1}{(z^2 + 1)(z^2 + 2z + 2)}\, dz$$

$$= 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2 + 1)(z^2 + 2z + 2)} + 2\pi i \operatorname*{Res}_{z=-1+i} \frac{1}{(z^2 + 1)(z^2 + 2z + 2)}$$

$$= 2\pi i \left[\frac{1}{2i(1 + 2i)} + \frac{1}{2i(1 - 2i)} \right] = \frac{2\pi}{5}.$$

8. (a)

$$\operatorname*{Res}_{z=n} \cot \pi z = \operatorname*{Res}_{z=n} \frac{\cos \pi z}{\sin \pi z} = \lim_{z \to n} \frac{\cos \pi z}{\pi \cos \pi z} = 1/pi$$

Using a square contour centred on the origin with sides of length $2N + 1$ it can be shown the as $N \to \infty$

$$\oint_C f(z) \cot \pi z\, dz \to 0$$

Hence sum of residues will be 0. $f(z)$ has poles at $z = \pm 2i$ so

$$0 = \operatorname*{Res}_{z=-2i} \frac{\cot \pi z}{z^2 + 4} + \operatorname*{Res}_{z=-2i} \frac{\cot \pi z}{z^2 + 4} + \sum_{n=-\infty}^{\infty} \operatorname*{Res}_{z=n} \frac{\cot \pi z}{z^2 + 4}$$

$$= \frac{\cot(-2\pi i)}{-4i} + \frac{\cot(2\pi i)}{4i} + \sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2 + 4)} = -\frac{\coth(2\pi)}{4} - \frac{\coth(2\pi)}{4} + \sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2 + 4)}$$

So

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi(n^2 + 4)} = \frac{\pi \coth(2\pi)}{2}$$

(b) Poles at $\pm 2i$. Taylor series for $|z| < 2$, and Laurent series for $2 < |z|$.

For $|z| < 2$

$$f(z) = \frac{z^2}{z^2 + 4} = \frac{z^2}{4(1 + z^2/4)} = \frac{z^2}{4} \left(1 - \left(\frac{z^2}{4} \right) + \left(\frac{z^2}{4} \right)^2 - \cdots \right) = \frac{z^2}{4} - \frac{z^4}{16} + \cdots.$$

For $|z| > 2$,

$$f(z) = \frac{z^2}{z^2 + 4} = \frac{1}{(1 + 4/z^2)} = 1 - \frac{4}{z^2} + \frac{16}{z^4} - \cdots.$$