1. (i) Find the area of the parallelogram whose vertices are

$$(4,5),\ (6,9),\ (7,10),\ (9,14)$$

(ii) Find the volume of the parallelepiped whose vertices are

$$(1,1,2),\ (2,1,2),\ (2,2,2),\ (1,2,3),\ (3,2,2),\ (2,2,3),\ (2,3,3),\ (3,3,3)$$

(iii) Define the Laplacian of a scalar function ϕ.

(iv) Evaluate the Laplacian of $z\sin(xy^2)$.

(v) Find the value of a such that there is a vector field $\mathbf{F}$ which satisfies

$$\mathrm{curl}(\mathbf{F}) = (1 - x^2, z^3, axz - \exp(y))$$

(vi) Find a scalar field ϕ such that

$$\mathrm{grad}(\phi) = (2x\cos(z^2), 3y^2\cos(z^2), -2z(x^2 + y^3)\sin(z^2))$$

(vii) Let S be the surface parametrised by

$$(r,\theta) \mapsto (r\sin\theta, r\cos\theta, r^3)$$

Find a normal vector to S at the point $(0,2,8)$.

2. Let $\mathbf{F}$ be the vector field $(x^2y, xz+y, (x^2+y^2)^{-3})$. Evaluate the line integral $\int_C \mathbf{F}.d\mathbf{s}$ along each of the following curves C from $(0,0,0)$ to $(2,4,0)$

(i) C is the curve parametrised by $t \mapsto (t, t^2, 0)$ for $0 \leq t \leq 2$.

(ii) C is the curve parametrised by $t \mapsto (2t, t^2 + 3t, 0)$ for $0 \leq t \leq 1$.

Let T be the part of the surface $x + 2y + 2z = 2$ in the first octant, $x \geq 0$, $y \geq 0$, $z \geq 0$.

(iii) Give a parametrisation of each of the three sides of the triangle T.

(iv) Give a parametrisation of the triangular surface T.

Turn over . . .

3. State Stokes' theorem.

Let S be the parametrised surface given by

$$(r, \theta) \mapsto (r \sin \theta, r \cos \theta, r^2)$$

for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

(a) Give a parametrisation of the boundary of S.

(b) Verify Stokes' theorem for the surface S and the vector field $\mathbf{F}$ defined by $\mathbf{F} = (z^2, -3xy, xy^3)$

4. (i) State the divergence theorem.

(ii) Using the divergence theorem give the co-ordinate free definition of the value of the divergence of a vector field $\mathbf{F}$ at a point $\mathbf{x}$.

(iii) Let S be the surface of the cube $-1 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 2$. Let $\mathbf{F}$ be the vector field $(\sin(yz), \exp(x^2 z), z)$. Use the divergence theorem to evaluate the flux integral

$$\int_S \mathbf{F}.d\mathbf{A}$$

(iv) Let S be the surface parametrised by

$$(r, \theta) \mapsto (ar \sin \theta, br \cos \theta, 0)$$

where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Find the area of S in terms of the constants a and b.

Turn over ...

5. Use contour integration techniques to evaluate the following integrals, stating clearly the contour used:

(i)
$$\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta}\, d\theta,$$

(ii)
$$\int_{-\infty}^{\infty} \frac{1}{(x^4 + 4x^2 + 3)}\, dx.$$

6. (a) Show that the singularities of $\operatorname{cosec} \pi z$ are located at $z = 0, \pm 1, \pm 2, \ldots$ and find their residues. Describe how you can use an integral of the form

$$\oint_C f(z) \operatorname{cosec} \pi z \, dz$$

to obtain the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} f(n)$$

for suitable $f(z)$. You should state the contour, C, to be used and give without proof the value of the above integral in the appropriate limit.

Use this result to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

You may quote the result that near $z = 0$

$$\operatorname{cosec} z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + \cdots$$

(b) Identify the poles of

$$f(z) = \frac{z}{z^2 - 1}.$$

This function has two different Taylor/Laurent series centred on $z = 0$ with different regions of validity. From the locations of the poles or otherwise identify what these two regions will be.

Find the Taylor/Laurent series for $f(z)$ in each of the two regions, showing terms involving z^n for $-5 \le n \le 5$.

7. Show that if $f(z) = u(x,y) + iv(x,y)$ is an analytic function then the Cauchy-Riemann equations linking the derivatives of $u(x,y)$ and $v(x,y)$ are satisfied.

   What test can be used to determine whether a function $u(x,y)$ could be the real part of an analytic function? Apply this test to the following two functions:

   (i) $u(x,y) = x^4 - 4x^2y^2 + y^4 + \tan^{-1}(y/x)$,

   (ii) $u(x,y) = x^3y - xy^3 + \tan^{-1}(x/y)$.

   For the function that could be the real part of an analytic function, find the corresponding conjugate harmonic function $v(x,y)$, and hence find the analytic function $f(z) = u(x,y)+iv(x,y)$ in terms of the complex variable z only.

8. A function $f(x)$ is periodic with period p. Write down its general Fourier expansion along with integral expressions for the coefficients a_0, a_n and b_n with $n \geq 1$.

   The periodic function $f(x)$ has period 2. It is defined in the interval $0 \leq x \leq 1$ by

$$f(x) = 4x(1-x).$$

   Sketch this function on the interval $-2 \leq x \leq 4$ for the two cases (i) $f(x)$ is an odd function, and (ii) $f(x)$ is an even function. What can be deduced about the Fourier coefficients in these two cases?

   Given that $f(x)$ is an *odd* periodic function, by using integration by parts when required, calculate the Fourier coefficients for the function $f(x)$. Hence write down its Fourier expansion.

   Deduce from this expansion that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

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