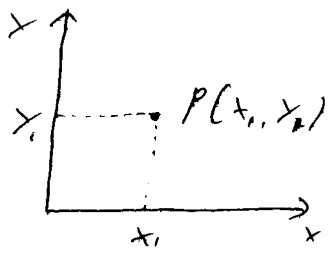
3) Analytic (coordinate) geometry (2D)

In this part of the course we study properties of curves and surfaces in two dimensional coordinate systems. (later 3D)

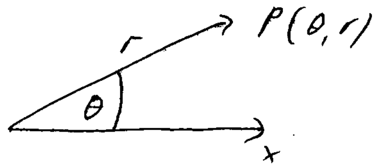
3.1) Points

Any point P in a two dimensional plane can be uniquely identified by a pair of coordinates in some reference frame. The frame can be Cartesian, polar, etc.

Cartesian coordinates



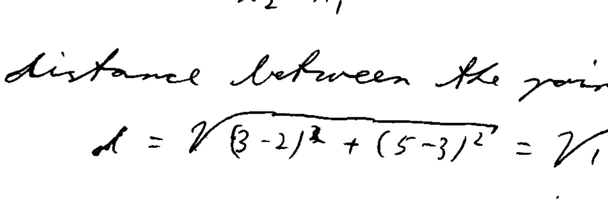
Polar coordinates



- The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

To see this just examine the diagram and use Pythagoras's theorem



$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

$$\Rightarrow \quad d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Expl.: What is the distance between the points $A(2,3)$ and $B(3,5)$?

$$d = \sqrt{(3-2)^2 + (5-3)^2} = \sqrt{1 + 2^2} = \sqrt{5}$$

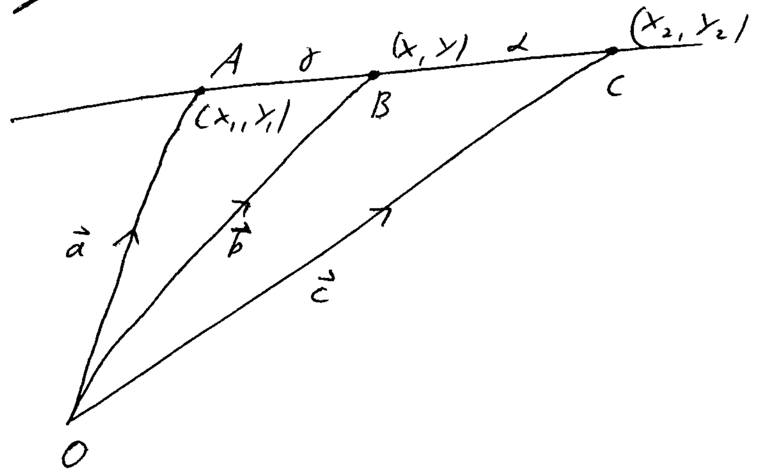
- The midpoint M between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is obviously $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

3.2) Lines

3.2.1) The equation of a straight line

We already know that two points are enough to specify a line, which means having three points must be an overdetermined system.

Consider the figure:



$$\overline{AB} : \overline{AC} = \gamma : (\alpha + \gamma) \quad \Rightarrow \quad \vec{AB} = \frac{\gamma}{\alpha+\gamma}\vec{AC} = \frac{\gamma}{\alpha+\gamma}(\vec{c} - \vec{a})$$

$$\Rightarrow \quad \vec{b} = \vec{a} + \vec{AB} = \vec{a} + \frac{\gamma}{\alpha+\gamma}(\vec{c} - \vec{a})$$

$$\Rightarrow \quad \underbrace{(\alpha + \gamma)}_{-\beta}\vec{b} = \vec{a}\alpha + \cancel{\vec{a}\gamma} + \gamma\vec{c} - \cancel{\gamma\vec{a}}$$

$$\Leftrightarrow \quad \vec{0} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$$

In components:

$$0 = \alpha x_1 + \beta x + \gamma x_2 \qquad (1)$$

$$0 = \alpha y_1 + \beta y + \gamma y_2 \qquad (2)$$

$$0 = \alpha + \beta + \gamma \qquad (3)$$

We can regard (1), (2) and (3) as three equations for the unknowns α, β and γ. This system has nonzero solutions if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x & x_2 \\ y_1 & y & y_2 \end{vmatrix} = 0$$

(see algebra course)

$$\Rightarrow \quad x y_2 - y x_2 - x_1 y_2 + x_2 y_1 + x_1 y - x y_1 = 0$$

$$\Rightarrow \quad y(x_1 - x_2) + x(y_2 - y_1) + (x_2 y_1 - x_1 y_2) = 0$$

$\Rightarrow$ equation of a line

$$\boxed{y + \frac{y_2 - y_1}{x_1 - x_2}\, x + \frac{x_2 y_1 - x_1 y_2}{x_1 - x_2} = 0}$$

In other words from two points we can determine all other points on the line.

Expl.: Determine the equation for the line passing through the points $A(-7, 2)$ and $B(3, -5)$:

$$y + \frac{-5-2}{-7-3}\, x + \frac{3 \cdot 2 - (-7)(-5)}{-7-3} = 0$$

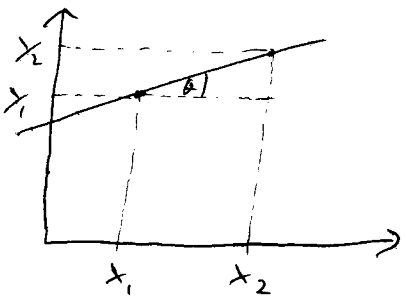
$$\Rightarrow \quad \underline{y = -\frac{7}{10}\, x + \frac{29}{10}}$$

This is a more familiar form with

$$y = \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{m}\, x + \underbrace{\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}}_{c}$$

with m being the gradient of the line (measuring the slope) and c the intercept with the y-axis.

From Expl. 3 in section 2.6. we know the projections



$$x_2 - x_1 = \cos\theta \quad ; \quad y_2 - y_1 = \sin\theta$$

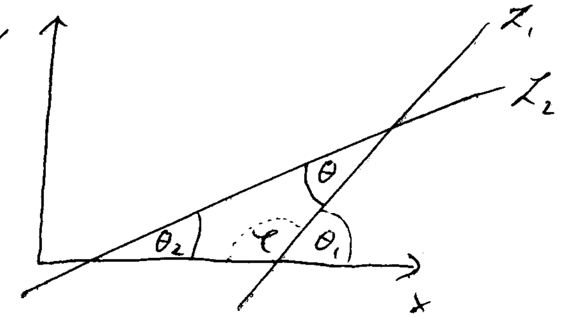
$$\Rightarrow \quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

3.2.2) The angle between two straight lines

Two lines L_1 and L_2 with gradients m_1 and m_2, respectively, intersect with an angle θ given by

$$\boxed{\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}}$$

Proof.:



from diagram:

$$\pi = \varphi + \theta + \theta_2$$

$$\pi = \varphi + \theta_1$$

$$\Rightarrow \quad \theta = \theta_1 - \theta_2$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

trigonometric identity

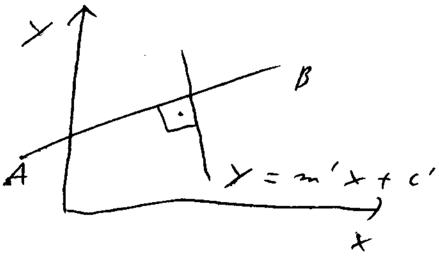
Expl 1) Parallel lines have the same gradient:

$$\theta = 0 \Rightarrow \quad \tan\theta = 0 \quad \Rightarrow \quad \theta_1 = \theta_2 \quad \Leftrightarrow \quad m_1 = m_2$$

Expl 2) Perpendicular lines have $m_1 m_2 = -1$

$$\theta = \frac{\pi}{2} \Rightarrow \quad \tan\theta \to \infty \quad \Rightarrow \quad 1 + m_1 m_2 \to 0 \quad \Rightarrow \quad m_1 m_2 = -1$$

Expl 3) Determine the equation of the line passing perpendicular through the midpoint of the line between $A(-3, 2)$ and $B(5, 6)$.



- The midpoint is $M\left(\frac{-3+5}{2}, \frac{2+6}{2}\right) = (1, 4)$
- The gradient of $\overline{AB}$ is $m = \frac{6-2}{5-(-3)} = \frac{4}{8} = \frac{1}{2}$

$\Rightarrow$ The slope of the line we want to find is $\underline{m' = -2}$

$\Rightarrow \quad Y = -2X + C'$ which has to hold for the midpoint M

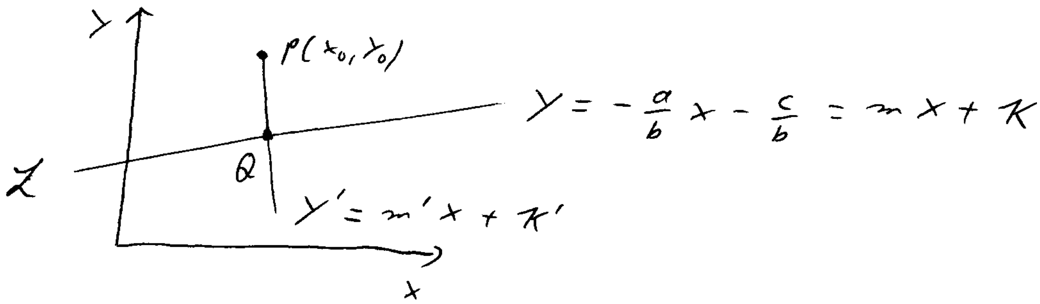
$\Rightarrow \quad 4 = -2 + c' \quad \Rightarrow \quad c' = 6 \quad \Rightarrow \quad \underline{Y = -2 + 6}$

3.2.3) The distance of a point from a line

The distance of a point $P(x_0, y_0)$ from a line $\mathcal{L}$ given by the equation $ax + by + c = 0$ is

$$\boxed{d = \left| \frac{a x_0 + b y_0 + c}{\sqrt{a^2 + b^2}} \right|}$$

Proof: (coursework 2)

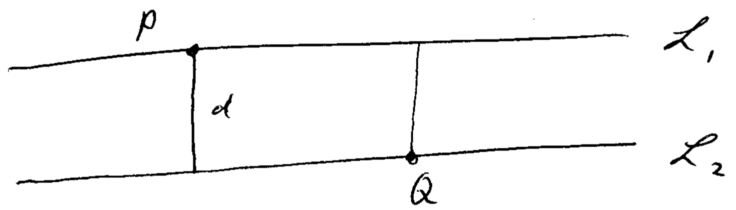


Expl 1) Find the distance d of the point $P(2,-1)$ to the line

$3x + 4y - 5 = 0$;

$$\Rightarrow \quad d = \left| \frac{3\cdot 2 + 4(-1) - 5}{\sqrt{3^2+4^2}} \right| = \left| \frac{6-9}{\sqrt{25}} \right| = \frac{3}{5}$$

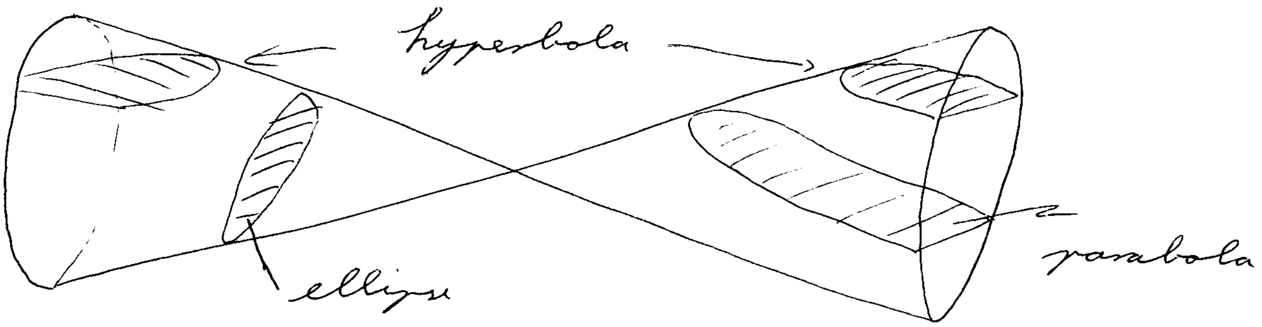
Expl 2) The distance between parallel lines is determined by computing the distance of any point on line 1 to line 2 using the previous formula. (recall axiom 5)



Obviously: distance P from L_2 = distance Q from L_1

3.3) Conic sections

Cut a cone with a plane



Here we do not derive these curves from 3D. We saw already that the solutions to the equation

$$f(x, y) = 0 \qquad \text{with } f(x,y) = ax + by + c$$

represents the equation of a line. Now we allow also quadratic terms in f, i.e. we consider

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Def.: A conic is a curve in a Euclidean plane which can be represented in polar coordinates (r, θ) by the equation

$$\boxed{0 = f(r, \theta) = r - \frac{k}{1 - e\cos\theta}} \qquad \begin{array}{l} e \geq 0 \\ k > 0 \end{array}$$

e is called the eccentricity and controls the shape of the curve. k is an overall constant.

We can transform this to Cartesian coordinates

$$r = k + r e \cos\theta$$

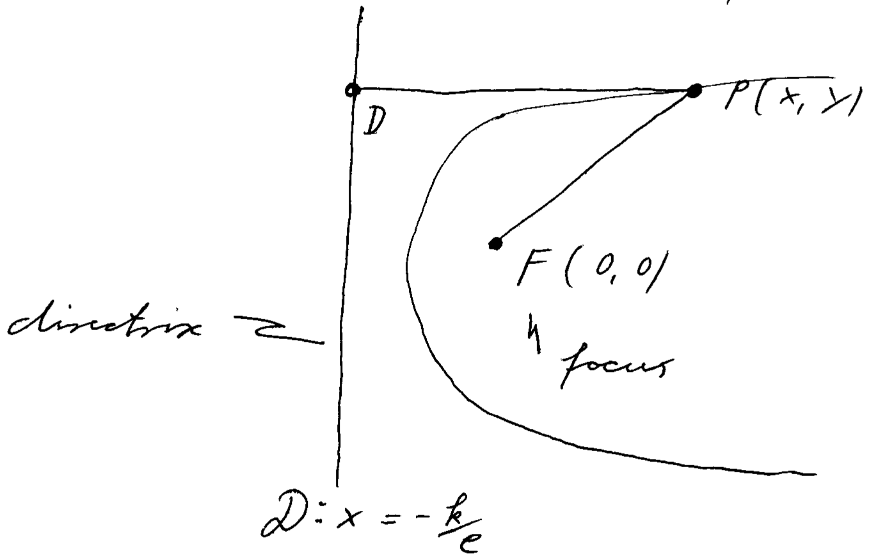
$$r^2 = (k + e r \cos\theta)^2$$

Now introduce $x = r\cos\theta$, $y = r\sin\theta$ such that

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{1} = r^2$$

$$\Rightarrow \quad x^2 + y^2 = (k + ex)^2 \quad \Leftrightarrow \quad f(r,\theta) = 0$$
$$= e^2\left(\frac{k}{e} + x\right)^2$$

Geometrical interpretation of this equation



$$\overline{FP} = \sqrt{x^2 + y^2}$$

$$\overline{DP} = x - \left(-\frac{k}{e}\right) = x + \frac{k}{e}$$

$$\Rightarrow \boxed{\overline{FP} = e\,\overline{DP}}$$

In other words a conic, that is the solutions to $f(r,\theta) = 0$, is described by all points P such that the distance to a fixed point F is a fixed ratio to the distance of P to a line $\mathcal{D}$, i.e. $e = \overline{FP} / \overline{DP} = \text{const.}$

Important to study planetary motion.

Depending on the values of e, conics can be classified:

3.3.1.) The Ellipse ($e < 1$)

We manipulate the equation

$$x^2 + y^2 = (k + ex)^2$$

$$\Rightarrow \quad (1 - e^2)x^2 - 2kex + y^2 = k^2 \qquad e \neq 1$$

$$x^2 - \frac{2ke}{1-e^2} + \frac{y^2}{1-e^2} = \frac{k^2}{1-e^2}$$

Complete the square:

$$\underbrace{x^2 - \frac{2ke}{1-e^2} + \left(\frac{ke}{1-e^2}\right)^2}_{\left(x - \frac{ke}{1-e^2}\right)^2} - \left(\frac{ke}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{k^2}{1-e^2}$$

$$\Rightarrow \quad (1-e^2)\left(x - \frac{ke}{1-e^2}\right)^2 + y^2 = k^2 + \frac{k^2 e^2}{1-e^2} = \frac{k^2}{1-e^2}$$

Now transform the variables:

$$x \longrightarrow X = x - \frac{ke}{1-e^2}$$

$$y \longrightarrow Y$$

$$\Rightarrow \quad \frac{(1-e^2)^2}{k^2} X^2 + \frac{1-e^2}{k^2} Y^2 = 1$$

$$\Rightarrow \quad \boxed{\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1}$$

which is a more familiar form (NORMAL FORM)

with $\underline{a = \frac{k}{1-e^2}}$, $\underline{b = \frac{k}{\sqrt{1-e^2}}}$ positive $\because$ $\underline{e < 1}$

In reverse, given a, b we can determine k and e:

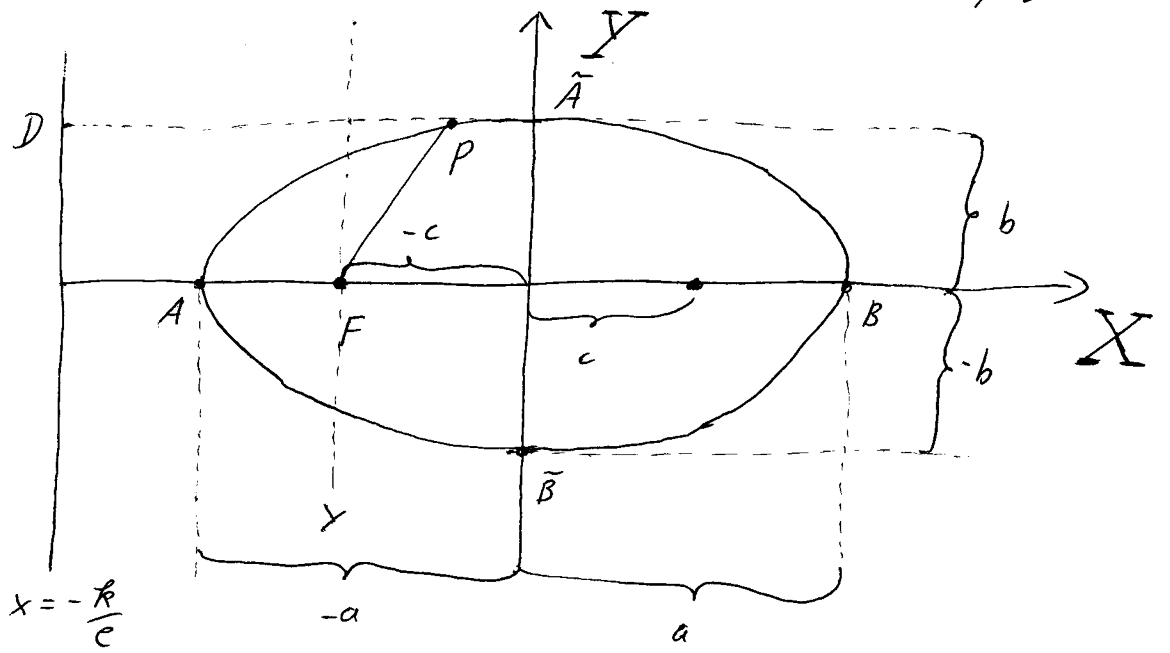
$$\frac{a^2}{b^2} = \frac{1}{1-e^2} \Rightarrow 1 - e^2 = \frac{b^2}{a^2} \Rightarrow \underline{e = \sqrt{1 - \frac{b^2}{a^2}}}$$

$$k = a(1-e^2) \Rightarrow \underline{k = \frac{b^2}{a}}$$

We shifted the focus by $\frac{ke}{1-e^2} =: c$

$$\Rightarrow \underline{c = ea}$$

We can collect all this information in a figure:



$\overline{FP} = e\,\overline{PD}$ $\quad$ $A, B \equiv$ vertex $\quad$ $F \equiv$ focus (foci at $\pm ea$) in $X\,Y$-system

$\overline{AB} \equiv$ major axis $= 2a$; $\quad$ $\tilde{A}\tilde{B} \equiv$ minor axis $= 2b$

<u>parameterisation:</u>

So far we considered $f(x, y) = 0$ or $f(r, \theta) = 0$ as static equations. We can also assume that x, y or r, θ are functions of some other variable, for instance t (time) or φ (some angle) and consider instead

$$f(x(t), y(t)) = 0 \quad \text{or} \quad f(r(t), \theta(t)) = 0$$

Such type of equations are called <u>parametric equations</u>. For the ellipse

$$X(\varphi) = a \cos \varphi \quad ; \quad Y(\varphi) = b \sin \varphi$$

or a rational parameterisation

$$X(t) = a\left(\frac{1-t^2}{1+t^2}\right) \quad ; \quad Y(t) = b\,\frac{2t}{1+t^2} .$$

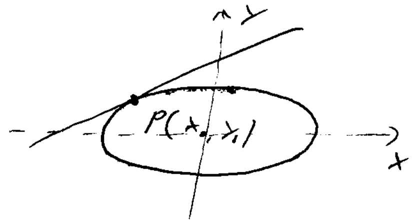
It is easy to see that these functions solve $\frac{X}{a^2} + \frac{Y}{b^2} = 1$.

Tangents:

Obviously lines can either cross the ellipse in two points, in no point at all or in just one point. Lines of the latter type are called tangents.

The tangent on an ellipse with centre at the origin at the point $P(x_0, y_0)$ is given by the equation

$$\boxed{\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1}$$



Proof: For the ellipse and the line to have a common point we must solve simultaneously

$$f_{ellipse}(x, y) = 0 \quad \text{and} \quad f_{line}(x, y) = 0$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad y = mx + c$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Leftrightarrow \quad \frac{x^2}{a^2} + \frac{m^2 x^2}{b^2} + \frac{2mcx}{b^2} + \frac{c^2}{b^2} - 1 = 0$$

$$\Leftrightarrow \quad \underbrace{\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right)}_{\alpha} x^2 + \underbrace{\frac{2mc}{b^2}}_{\beta} x + \underbrace{\left(\frac{c^2}{b^2} - 1\right)}_{\gamma} = 0$$

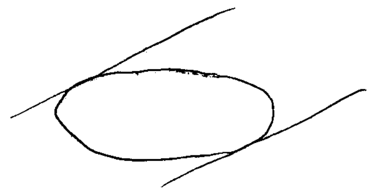
$\Rightarrow$ solutions: $x_{1/2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$

to have only one solution means $x_1 = x_2 \Leftrightarrow \beta^2 = 4\alpha\gamma$

$$\Leftrightarrow \quad \cancel{\frac{4m^2c^2}{b^4}} = 4\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)\left(\frac{c^2}{b^2}-1\right) = 4\left(\frac{c^2}{a^2b^2}-\frac{1}{a^2}+\cancel{\frac{m^2c^2}{b^4}}-\frac{m^2}{b^2}\right)$$

$$\Rightarrow \quad c^2 = b^2 + a^2 m^2$$

$$\Rightarrow \quad \underline{Y_{1/2} = m X \pm \sqrt{b^2+a^2m^2}}$$



$\underset{\sim}{m}$? $P(x_0, y_0)$ is on the line and the ellipse:

$$(1) \quad \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \qquad \text{and} \qquad y_0 = m x_0 \pm \sqrt{b^2+a^2m^2} \quad (2)$$

$$\Rightarrow (2)^2: \quad b^2 + a^2m^2 = (y_0 - m x_0)^2 = y_0^2 - 2 m x_0 y_0 + m^2 x_0^2$$

$$\Rightarrow \quad \underbrace{(x_0^2 - a^2)}_{-\frac{a^2}{b^2} y_0^2 \text{ with } (1)} m^2 - 2 x_0 y_0 m + \underbrace{(y_0^2 - b^2)}_{-\frac{b^2}{a^2} x_0^2 \text{ with } (1)} = 0$$

$$\Rightarrow \quad \frac{a^2}{b^2}\frac{y_0}{x_0} m^2 + 2m + \frac{b^2}{a^2}\frac{x_0}{y_0} = 0$$

$$\Rightarrow \quad m_{1/2} = \frac{-2 \pm \sqrt{4 - 4 \frac{a^2}{b^2}\frac{y_0}{x_0}\frac{b^2}{a^2}\frac{x_0}{y_0}}}{2 \frac{a^2}{b^2}\frac{y_0}{x_0}} = -\frac{b^2 x_0}{a^2 y_0}$$

(this is what we expected, just one m)

$$\Rightarrow \quad Y = -\frac{b^2}{a^2}\frac{x_0}{y_0} X \pm \sqrt{b^2 + a^2 \frac{b^4 x_0^2}{a^4 y_0^2}} \qquad \Big| \cdot \frac{y_0}{b^2}$$

$$\Rightarrow \quad \frac{y_0 Y}{b^2} + \frac{x_0 X}{a^2} = \pm \frac{y_0}{b^2}\sqrt{b^2 + \frac{b^4 x_0^2}{y_0^2 a^2}} = \pm 1$$

with (1)

the minus sign can be excluded as $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} > 0$ q.e.d.

3.3.2) The hyperbola ($e > 1$) → Expl.:

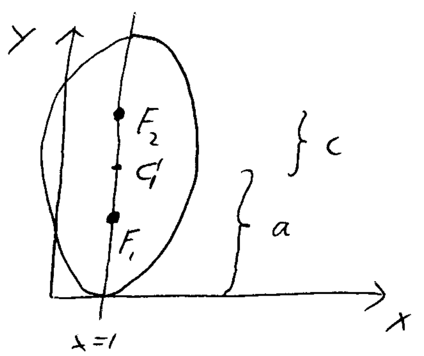
In the same way as for the ellipse we derive from $f(r,\theta) = 0$

$$\underbrace{\frac{(1-e^2)^2}{k^2}}_{>0} X^2 + \underbrace{\frac{(1-e^2)}{k^2}}_{<0} Y^2 = 1$$

but now

Expl.: Find the equation of the ellipse with foci at $F_1(1,2)$ and $F_2(1,6)$. The major axis has length 8. Find its eccentricity.

- X-coordinate of foci does not change ⇒ major axis is along the y-axis at $x=1$
- $F_1(1,2)$, $F_2(1,6)$ ⇒ centre at $C=(1,4)$
- length of major axis is 8 ⇒ Vertices at $A(1,0)$, $A'(1,8)$



- so far we know: $\dfrac{(x-1)^2}{b^2} + \dfrac{(y-4)^2}{a^2} = 1$

• notice $a \leftrightarrow b$ because major axis is along Y

- $a = 4$ (major axis length/2)

$c = e\,a = 2 \qquad e = \sqrt{1 - b^2/a^2}$

$\Rightarrow \quad \left(1 - \frac{b^2}{a^2}\right) a^2 = 4 \quad \Rightarrow \quad a^2 - b^2 = 4 \quad \overset{a=4}{\Rightarrow} \quad b^2 = 12$

$$\Rightarrow \quad \boxed{\frac{(x-1)^2}{12} + \frac{(y-4)^2}{16} = 1}$$

$$\Rightarrow \quad \underline{\underline{e}} = \sqrt{1 - \frac{12}{16}} = \sqrt{\frac{1}{4}} = \underline{\underline{\frac{1}{2}}}$$

Therefore the equation takes on the form

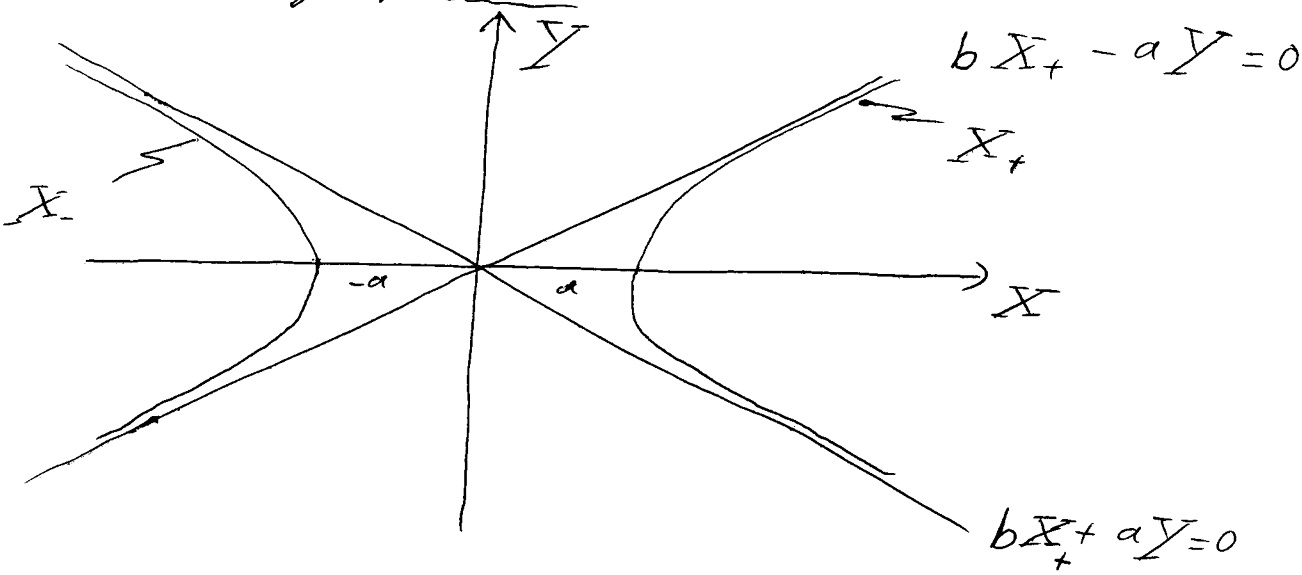
$$\boxed{\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1}$$ with $a, b > 0$

and $a = \frac{k}{1-e^2}$; $b = \frac{k}{\sqrt{e^2-1}}$

Solve for e and k instead: $\frac{a^2}{b^2} = \frac{1}{e^2-1} \Rightarrow e = \sqrt{1+b^2/a^2}$

$$k = a(1-e^2) = -\frac{b^2}{a} = k$$

parameterisation (asymptotes):



$X_\pm = \pm a \cosh \varphi$ $\qquad$ $Y = b \sinh \varphi$

Consider $\lim_{\varphi \to \pm\infty} (X_+, Y) \to (+\infty, \pm\infty)$

$$bX_+ - aY = ab\cosh\varphi - ab\sinh\varphi = ab\, e^{-\varphi}$$

$$\Rightarrow \lim_{\varphi\to\infty} (bX_+ - aY) = 0$$

$$bX_+ + aY = ab\cosh\varphi + ab\sinh\varphi = ab\, e^{\varphi}$$

$$\Rightarrow \lim_{\varphi\to-\infty} (bX_+ + aY) = 0$$

Thus the point (X_+, Y) approaches the asymptote $bX_+ - aY = 0$ for large $+\varphi$ and the asymptote $bX_+ + aY = 0$ for large $(-\varphi)$.

A similar argument holds for the other branch.

3.3.3) The Parabola ($e=1$)

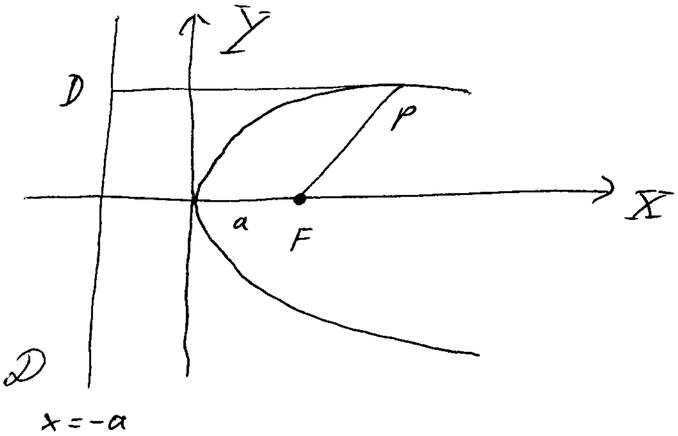
Now $\quad x^2+y^2=(k+x)^2 = k^2+2kx+x^2$

$$\Rightarrow \quad y^2 = 2kx + k^2$$

Transform $\quad x \to X - \frac{1}{2}k$

$y \to Y \qquad\qquad k = 2a$

$$\Rightarrow \quad \boxed{Y^2 = 4aX} \qquad \text{normal form}$$



parameterisation: $\quad X = at^2 \qquad Y = 2at$

tangents: The tangent on a parabola $Y^2 = 4ax$ at the point $P(x_0, y_0)$ is given by the equation

$$\boxed{Y = \frac{y_0}{2x_0}x + \frac{2x_0}{y_0}a}$$

Proof: We have to solve $y^2=4ax$, $y = mx+c$ simultaneously.

$$\Rightarrow \quad m^2x^2 + 2mcx + c^2 = 4ax$$

$$\Leftrightarrow \quad \underbrace{m^2}_{\alpha}x^2 + \underbrace{(2mc-4a)}_{\beta}x + \underbrace{c^2}_{\gamma} = 0$$

$$\Rightarrow \quad x_{1/2} = \frac{-\beta \pm \sqrt{\beta^2-4\alpha\gamma}}{2\alpha} \quad \Rightarrow x_1 = x_2 \Leftrightarrow \beta^2 = 4\alpha\gamma$$

$$\Rightarrow \quad 4m^2c^2 + 16a^2 - 16mca = 0 \quad \Rightarrow \underline{c = a/m}$$

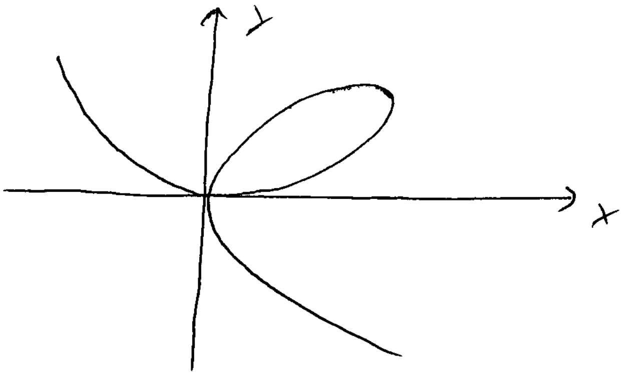
At $P(x_0,y_0)$: $\quad y_0 = mx_0 + a/m \Leftrightarrow m^2x_0 - my_0 + a = 0 \Rightarrow \underline{m = \frac{y_0}{2x_0}}$

3.4) Curves of degree > 2

We just present here some examples

3.4.1) Descartes folium

$$\boxed{f(x,y) = x^3 + y^3 - 3xy = 0}$$



(Descartes did not understand negative coordinates)

3.4.2) Fermat curves

$$\boxed{f(x,y) = x^n + y^n - 1 = 0}$$

Is it possible to solve this equation with x, y being rational?

$n=2$: $\quad x = \frac{a}{c}, \quad y = \frac{b}{c} \quad \Rightarrow \quad a^2 + b^2 = c^2$

solution are the Pythagorean triads, e.g.

$a = 3, \quad b = 4, \quad c = 5$

$a = 5, \quad b = 12, \quad c = 13$

$a = 65, \quad b = 72, \quad c = 97$

$n \geq 3$: It is not possible to find rational solutions to the equation $x^n + y^n = 1$ for $n \geq 3$. (Fermat's last theorem)

Proof: only in 1995 (A. Wiles)