E.Ivanov + A.S.; hep-th/0703038

CRYPTOREALITY OF NONANTICOMMUTATIVE HAMILTONIANS

Noncommutative theories: $[x_\mu, x_\nu] = \omega_{\mu\nu}$

Star Product:

$$f(x)g(x) \to f(x)\star g(x) = \exp\left\{-\frac{\omega_{\mu\nu}}{2}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\right\} f(x)g(y)\Big|_{x=y}$$

then

$$x_\mu \star x_\nu - x_\nu \star x_\mu = \omega_{\mu\nu}$$

- NC Lagrangians

$$\lambda \int \phi^3(x)\, d^n x \ \to \lambda \int \phi \star \phi \star \phi \, d^n x$$

- Superspace: $(x_\mu, \theta_\alpha, \bar\theta^{\dot\alpha})$, grassmanian $\theta_\alpha, \bar\theta^{\dot\alpha}$.
- superfields: $\Phi(x_\mu, \theta_\alpha, \bar\theta^{\dot\alpha})$
- SUSY Lagrangians: $S = \int d^4x d^2\theta d^2\bar\theta F(\Phi_i)$

NAC theories: $\{\theta, \theta\} \neq 0$

Example: NAC Wess-Zumino model

(Seiberg, 03)

- deformation:

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} , \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = \{\theta^\alpha, \bar\theta^{\dot\beta}\} = 0$$

- star product:

$$X(\theta) \star Y(\theta) \;=\; \exp\left\{-\frac{C^{\alpha\beta}}{2}\frac{\partial^2}{\partial\theta_1^\alpha \partial\theta_2^\beta}\right\} X(\theta_1)Y(\theta_2)\Big|_{1=2}$$

- Lagrangian:

$$\mathcal{L} \;=\; \int d^4\theta\, \bar{\Phi} \star \Phi + \left[\int d^2\theta \left(\frac{m\Phi \star \Phi}{2} + \frac{\lambda\Phi \star \Phi \star \Phi}{3}\right) + \mathrm{c.c}\right]$$

- only the last term is deformed

$$\frac{\lambda}{3}\int d^2\theta\, \Phi^3 \;\to\; \int d^2\theta\, \Phi * \Phi * \Phi = \lambda F \phi^2 - \frac{\lambda}{3}\, \det \|C\| F^3 \;.$$

- Lorentz invariant, but complex

Does it mean anything in real time ?

YES !

• The Hamiltonian is Hermitian in disguise (cryptoreal).

Cryptoreal Hamiltonians

(M.Gasymov, 80 ($V(x) = e^{ix}$); C.Bender + S.Boettcher, 98; A.Mostafazadeh, 05)

Example:

$$H = \frac{p^2 + x^2}{2} + igx = \frac{p^2 + x'^2}{2} + \frac{g^2}{2}$$

with $x' = x + ig = e^{-gp} x e^{gp}$. Then $\tilde{H} = e^{-gp} H e^{gp}$ is Hermitian.

Example:

$$H = \frac{p^2 + x^2}{2} + igx^3 \ .$$

Take

$$R = g\left(\frac{2}{3}p^3 + x^2 p\right) - g^3\left(\frac{64}{15}p^5 + \frac{20}{3}p^3x^2 + 4px^4 - 6p\right) + O(g^5) \ .$$

Then

$$\tilde{H} = e^R H e^{-R} = \frac{p^2 + x^2}{2} + g^2\left(3p^2x^2 + \frac{3x^4}{2} - \frac{1}{2}\right) + O(g^4)$$

is Hermitian.

- New coord. and momenta:

$$p' = e^R p e^{-R} = p + 2igxp + g^2(2p^3 - px^2) + \ldots$$
$$x' = e^R x e^{-R} = x - ig(x^2 + 2p^2) - g^2(x^3 - 2xp^2) + \ldots \ .$$

• $\{p', x'\}_{P.B.} \neq 1$ starting from the terms $\sim g^4$.
Not a canonical transformation !

• Weyl symbols:

$$xp \longrightarrow (\hat{p}x + x\hat{p})/2,$$
$$x^2p \longrightarrow (x^2\hat{p} + \hat{p}x^2 + x\hat{p}x)/3\ , \ldots$$

Example: $H = \bar{\pi}\pi + \bar{z}z + gz^3$
Take

$$R = -ig\left(\bar{\pi}z^2 + \frac{2}{3}\bar{\pi}^3\right)\ .$$

Then $\tilde{H} = e^R H e^{-R} = \bar{\pi}\pi + \bar{z}z$.

• Holomorphic deformation is rotated away without trace !

NAC SQM

(Aldrovandi + Schaposnik, 06)

$$S \;=\; -\int dt\, d\bar{\theta} d\theta \left[\frac{1}{2}(D\star X)\star(\bar{D}\star X) + V_\star(X)\right],$$

- X - real supervariable
- $\star$ corresponds to deformations

$$\{\theta,\theta\} = C, \quad \{\bar{\theta},\bar{\theta}\} = \bar{C}, \quad \{\theta,\bar{\theta}\} = \tilde{C}$$

- $V_\star(X) = \sum_n c_n (X\star\ldots\star X)_n$.
- Lifting deformations $\rightarrow$ Witten's SQM in chiral basis,

$$t = \tau - i\theta\bar{\theta}, D = \frac{\partial}{\partial\theta} - 2i\bar{\theta}\frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial\bar{\theta}}$$

Lagrangian in components.

$$L \;=\; -i\dot{x}F - \frac{\partial \tilde{V}(x,F)}{\partial x}F + \frac{1}{2}F^2 + i\bar{\psi}\dot{\psi} + \frac{\partial^2 \tilde{V}(x,F)}{\partial x^2}\bar{\psi}\psi \,,$$

with

$$\tilde{V}(x,F) \;=\; \int_{-1/2}^{1/2} d\xi\, V(x+\xi c F) \;, \qquad\qquad c^2 = \tilde{C}^2 - C\bar{C} \;.$$

(cf. Alvarez-Gaume + Vazquez-Mozo, 05)

- Lagrangian and Hamiltonian are complex.
- Superchargres:

$$Q = \frac{\partial}{\partial\theta}, \quad \bar{Q} = -\frac{\partial}{\partial\bar{\theta}} - 2i\theta\frac{\partial}{\partial t}$$

- $\star$ product commutes with Q, but not with $\bar{Q}$.

- Only one obvious Nöther supercharge

$Q = \psi p$

- But

$$\bar{\mathcal{Q}} \; = \; \bar{\psi} \left(p + 2i \frac{\partial \tilde{V}}{\partial x} \right) .$$

also commutes with the Hamiltonian !

- Algebra

$$Q^2 = \bar{\mathcal{Q}}^2 = 0, \; \{Q, \bar{\mathcal{Q}}\} = 2H$$

holds

Cryptoreality of H

Let $V(X) = \lambda X^3/3$. Then $\tilde{V}(x, F) = \lambda x^3/3 + \lambda c^2 x F^2/12$ and

$$H = p^2/2 + i\lambda p x^2 - i\beta p^3 - 2\lambda x \bar{\psi}\psi \ ,$$ with $\beta = \lambda c^2/12$

- Rotate it with

$$R = -\lambda x^3/3 + \beta x p^2 - 2\lambda\beta x^2 \bar{\psi}\psi - 2\beta^2 p^2 \bar{\psi}\psi + O(\lambda^3, \beta^3, \lambda^2\beta, \lambda\beta^2) \ .$$

- The conjugated Hamiltonian

$$\tilde{H} = e^R H e^{-R} = p^2/2 - 2\lambda x \bar{\psi}\psi + [\lambda^2 x^4 + 3\beta^2 p^4]/2 + \lambda\beta/2 + \ldots$$

is Hermitian.

- rotated supercharges:

$$\tilde{Q} = e^{\hat{R}} Q e^{-\hat{R}} = \psi[p - i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \ldots]\,,$$
$$\tilde{\bar{Q}} = e^{R} \bar{\mathcal{Q}} e^{-R} = \bar{\psi}[p + i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \ldots]\,.$$

are adjoint to each other.

Field Theories

• deformed WZ. Extra holomorphic piece $\sim F^3$ is rotated away without trace ! Physically equivalent to undeformed theory.

NAC $\mathcal{N} = 2$ SQED

.

• deformation:

$$\{\theta_i^\alpha, \theta_j^\beta\} \; = \; \frac{1}{4} J \epsilon^{\alpha\beta} \epsilon_{ij} \; ,$$

• Lorentz symmetry is preserved.

- Lagrangian in components

(Ferrara et al, 05 Buchbinder et al, 06)

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_\phi + \mathcal{L}_\Psi + \mathcal{L}_A \,, \\
\mathcal{L}_\phi &= -\frac{1}{2}\Box\bar\phi\left[\phi + \frac{JA_mA_m}{1+J\bar\phi} + \frac{1}{4}\frac{J^3\partial_m\bar\phi\partial_m\bar\phi}{1+J\bar\phi}\right], \\
\mathcal{L}_\Psi &= i\left[\Psi^{i\alpha} + \frac{JA_m(\sigma_m)^{\alpha}_{\dot\alpha}\bar\Psi^{i\dot\alpha}}{1+J\bar\phi}\right](\sigma_n)_{\alpha\dot\beta}\,\partial_n\left(\frac{\bar\Psi^{\dot\beta}_i}{1+J\bar\phi}\right), \\
\mathcal{L}_A &= \frac{1}{4}(1+J\bar\phi)^2\left(f_{mn}f_{mn} + f_{mn}\tilde f_{mn}\right), \\
f_{mn} &= \partial_m\left(\frac{1}{1+J\bar\phi}A_n\right) - \partial_n\left(\frac{1}{1+J\bar\phi}A_m\right).
\end{aligned}
$$

- complex, but probably cryptoreal.

QM limit

$$H = 2P\bar{P} + \frac{1}{2}\,\vec{E}\vec{E} + \frac{2}{3}J^4\frac{P^4}{(1+J\bar{\phi})^2} - 2J^2A_0^2\frac{P^2}{(1+J\bar{\phi})^2}\,.$$

- Rotating it with

$$R = -\frac{i}{3}\frac{J^3P^3}{1+J\bar{\phi}} + iA_0^2\frac{JP}{1+J\bar{\phi}}\,,$$

we obtain undeformed quadratic Hamiltonian

$$H = 2P\bar{P} + \frac{1}{2}\,\vec{E}\vec{E}\,.$$

- Our bet: deformation of $\mathcal{N}=2$ SQED field theory gives a nontrivial interacting theory with cryptoreal Hamiltonian.