Quasi-hermitian Liouville Theory

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I will briefly discuss properties of quasi-hermitian theories in the context of a simple exactly solvable example, "imaginary" Liouville theory. I will begin with a deformation quantization approach to the point particle theory, i.e. QM in phase space. Then I will discuss the field theory extension of this model, of interest in string theory.

On-line version at *http://www.physics.miami.edu/~curtright/LondonPT2007.pdf*

Introduction

Superficially non-hermitian Hamiltonian quantum systems are of considerable current interest, especially in the context of PT symmetric models [4, 20], although many of the main ideas appeared earlier [24, 29]. For such systems the Hilbert space structure is at first sight very different than that for hermitian Hamiltonian systems inasmuch as the dual wave functions are *not* just the complex conjugates of the wave functions, or equivalently, the Hilbert space metric is *not* the usual one. While it is possible to keep most of the compact Dirac notation in analyzing such systems, in this talk I will mostly work with explicit functions and avoid abstract notation, in the hope to fully expose all the structure, rather than to hide it.

My discussion is focussed on a system – imaginary Liouville QM – with the simple potential

$$V(x) = \exp(2\,\boldsymbol{i}\,x)$$

This model, as well as its field theory extension, is of interest for applications to table-top physical systems [2, 5] and to deeper problems in string theory [25, 27]. I will not pursue those applications here, but rather I will discuss the elementary quantum mechanics of the point particle model, and then make a fairly straightforward extension of the point-particle results to the field theory. I believe this will be helpful in understanding the applications cited, as well as others.

Personal motivation and history ...

I became interested in quasi-hermitian models about two and one half years ago, upon hearing a talk on PT-symmetric theories by the P T Barnum of the subject (a.k.a. Carl Bender), at the first in the new series of *Miami winter conferences*.

Along with Luca Mezincescu, I considered (in perhaps excessive detail) the potential $V(x) = \exp(2\,\boldsymbol{i}\,x)$ to understand what all the fuss was about [10]. Our philosophy was *exactly solvable cases are best*, if you can find some, to illustrate interesting structure.

With two graduate students – David Schuster and Andre Veitia – we pursued related projects, including supersymmetric models [11], models analyzed by methods of deformation quantization [12], and extensions to field theories [all to appear in J Math Phys ... eventually].

With Evgeny Ivanov and Paul Townsend , we also explored Landau models on various supermanifolds using the biorthogonal techniques that we learned from the other studies [9].

A more compelling motivation ...

A field theory extension of the quasi-hermitian $V = \exp(2ix)$ potential model is Liouville field theory for imaginary coupling b.

$$\mathcal{L} = \frac{1}{4\pi} (\partial\phi)^2 - \mu \exp(2b\phi)$$

The Liouville model has been of interest since Polyakov's work in the early 1980s, and its quantum properties have been studied in depth by many people during the last 25 years. The theory has a central charge $c = 1 + 6Q^2$, where the background charge is given by $Q = (b + b^{-1})$. The conformal dimension of $e^{2a\phi}$ is $a\,(Q - a)$. So, both $e^{2b\phi}$ and $e^{2\phi/b}$ have conformal dimension 1, a point exploited by O'Raifeartaigh et al. [21]

Taking all this at face value, $b = i$ is a $Q = 0$, $c = 1$ conformal field theory. Other imaginary b give $c < 1$, perhaps negative.

Exact results have been obtained for the model using CFT techniques, especially the three-point function $\langle \exp(a_1\phi(x_1)) \exp(a_2\phi(x_2)) \exp(a_3\phi(x_3))\rangle$, as skillfully exploited by Dorn and Otto, and the brothers Zamolodchikov in the early 1990s. However ...

> "Due to the *non*existence of a SL(2,C) invariant vacuum one has to be careful with respect to the usual conformal structure of N-point functions."
>
> "Altogether we find a drastic change in the analytic structure $\cdots$ in going from $\mathrm{Re}\, b^2 > 0$ to $\mathrm{Re}\, b^2 < 0$."

– Dorn and Otto.

The complex model is of interest in string theory for the "rolling tachyon" problem. In fact, Andy Strominger et al., and also Volker Schomerus, have already discussed the exact solution for the three-point functions for $b \to i\beta$, $c \leq 1$ Liouville theory on the sphere, again using CFT techniques. In particular, Schomerus found that,

> "In the regime $c \leq 1$, Liouville theory does not depend smoothly on the central charge. Such a behavior is in sharp contrast to the properties of Liouville theory for $c > 1$."

Perhaps a quasi-hermitian Hamiltonian approach will shed light on these issues.

General theory

Many theories are "*quasi-hermitian*" as given by the entwining relation

$$GH = H^{\dagger}G$$

where "the metric" G is an hermitian, invertible, and *positive-definite* operator. All adjoints here are specified in a *pre-defined* Hilbert space, with a given scalar product and norm.

Existence of such a G is a necessary and sufficient condition for a completely diagonalizable H to have real eigenvalues. In such situations, it is *not* necessary that $H = H^{\dagger}$ to yield real energy eigenvalues.

Although – not to make it sound too simple – in infinite dimensional cases there are subtle issues about boundedness of the operators, and their domains [24]. Also, an interesting situation arises if G is positive but not positive-definite on the full Hilbert space.

Given H there are two widely-used methods to find all such G:

(I) Solve the entwining relation directly (e.g. as a PDE),

or

(II) Solve for the eigenfunctions of H, find their biorthonormal dual functions,

then construct $G \sim (\text{dual})^\dagger \times (\text{dual})$, or $G^{-1} \sim (\text{state}) \times (\text{state})^\dagger$.

In principle, these methods are equivalent. In practice, one or the other may be easier to implement.

Once you have a G, an equivalent hermitian Hamiltonian is

$$\mathbb{H} \quad = \quad \sqrt{G}\, H\, \sqrt{G^{-1}} = \mathbb{H}^{\dagger}$$

So why consider apparently non-hermitian structures at all?

A priori you may not know that G exists, let alone what it actually is. But even when you do have G, and finally $\mathbb{H}$, the manifestly hermitian form of an interesting model may be *non-local* and more difficult to analyze than an equivalent, local, quasi-hermitian form of the model.

In particular, non-locality readily appears for imaginary Liouville theory, as I will explain.

Alternatively, we may *define a new scalar product*, $\langle\cdot,\cdot\rangle_G$, differing from the original one $\langle\cdot,\cdot\rangle$, under which *the original Hamiltonian is* now *seen to be hermitian.*

$$\langle\alpha,\beta\rangle_G \equiv \langle\alpha, G\beta\rangle$$

$$\langle\alpha, H\beta\rangle_G \equiv \langle\alpha, GH\beta\rangle = \left\langle\alpha, H^\dagger G\beta\right\rangle \equiv \langle H\alpha,\beta\rangle_G = \langle\beta, H\alpha\rangle_G^*$$

Note that $\langle\cdot,\cdot\rangle_G$ is *positive definite*, so it can be used to construct a new norm for the vector space.

$$\|\alpha\|_G = \sqrt{\langle\alpha,\alpha\rangle_G} = \sqrt{\langle\alpha, G\alpha\rangle}$$

Exercise: Check completeness of the space using this new norm, to make sure that we indeed have a Hilbert space using $\langle\cdot,\cdot\rangle_G$. If so, then all the standard arguments for hermitian operators now go through using this new scalar product. It follows that the eigenvalues of H are real, its non-degenerate eigenvectors are orthogonal, etc.

In this way of thinking about the problem, H originally *appeared to be non-hermitian* only *because* our initial *choice of the* $\langle\cdot,\cdot\rangle$ *scalar product was somewhat misguided.* Non-hermiticity of H was illusory, not actual.

If there are any non-experts in the audience (which is highly doubtful at this conference!) I invite you to consider simple matrix examples to flesh out these ideas. For instance, the 2×2 Hamiltonian

$$H = \begin{pmatrix} 1 & i \sin\theta \\ i \sin\theta & -1 \end{pmatrix}$$

nicely illustrates the general theory. (See p 64, T Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.)

I now consider these abstract results in explicit detail for imaginary Liouville QM. (Classical dynamics for this system is discussed in Appendix 1 of my supplementary on-line notes.) It turns out to be convenient, in my opinion, to use a phase-space approach to the problem. (If QM in phase space is not familiar to you, please see my book with David Fairlie and Cosmas Zachos [30], or the quick tutorial in Appendix 2 of my on-line notes.)

Other notable applications of QMPS methods to PT symmetric models have been made by Scholtz and Geyer [22, 23], and by de Morisson Faria and Fring [14].

Entwining the metric in phase space

The previous entwining relation $GH = H^{\dagger}G$ or alternatively $HG^{-1} = G^{-1}H^{\dagger}$ can be written as a PDE through the use of deformation quantization techniques in phase space. If the Weyl kernel of G^{-1} is denoted by "the dual metric" $\widetilde{G}(x,p)$

$$G^{-1}(\mathsf{x},\mathsf{p}) = \frac{1}{(2\pi)^2}\int d\tau d\sigma dx dp\ \widetilde{G}(x,p)\exp(i\tau(\mathsf{p}-p)+i\sigma(\mathsf{x}-x))$$

then the entwining equation is

$$H(x,p)\star\widetilde{G}(x,p) \quad = \quad \widetilde{G}(x,p)\star\overline{H(x,p)}$$

where the associative *Groenewold star product* operation is ($\hbar \equiv 1$)

$$\star \quad \equiv \quad \exp\left(\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}-\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)$$

For the imaginary Liouville example

$$H(x,p) = p^2 + \exp(2ix) \quad , \qquad \overline{H(x,p)} = p^2 + \exp(-2ix)$$

and $H \star \widetilde{G} = \widetilde{G} \star \overline{H}$ boils down to the linear differential-difference equation

$$p \, \frac{\partial}{\partial x} \widetilde{G}(x,p) = \sin(2x) \;\; \widetilde{G}(x,p-1)$$

Hermitian G^{-1} is represented here by a *real* Weyl kernel $\widetilde{G}$.

Basic solutions of the metric equation

Basic solutions to the $H \star \widetilde{G}$ entwining relation are obtained by *separation of variables*. We find two classes of solutions, labeled by a parameter s. The *first class* of solutions is non-singular for all real p, although there are zeroes for negative integer p.

$$\widetilde{G}\left(x,p;s\right) = \frac{1}{s^{p}\Gamma\left(1+p\right)} \exp\left(-\frac{1}{2}s\cos 2x\right)$$

For real s this is real and positive definite on the positive momentum half-line.

Solutions in the *other class* have poles and corresponding changes in sign for positive p..

$$\widetilde{G}_{\text{other}}\left(x,p;s\right) = \frac{\Gamma\left(-p\right)}{s^{p}} \exp\left(\frac{1}{2}s\cos 2x\right)$$

Linear combinations of these are also solutions of the linear entwining equation.

This linearity permits us to build a *composite metric* from members of the first class by using a contour integral representation. Namely

$$\widetilde{G}(x,p) \equiv \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \widetilde{G}(x,p;s)\ \frac{e^{s/2}}{s}\ ds$$

The contour begins at $-\infty$, with $\arg s = -\pi$, proceeds below the real s axis towards the origin, loops in the positive, counterclockwise sense around the origin (hence the $(0+)$ notation), and then continues above the real s axis back to $-\infty$, with $\arg s = +\pi$.

Evaluation of the contour integral gives

$$\widetilde{G}(x,p) = \frac{\left(\sin^2 x\right)^p}{\left(\Gamma(p+1)\right)^2}$$

where we have made use of Sonine's contour representation of the Γ function.

$$\frac{1}{\Gamma(1+p)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \tau^{-p-1}e^{\tau}d\tau$$

The $\bigstar$ root of the metric

$\widetilde{S}$ as a direct solution of an entwining equation

We look for an equivalence between the Liouville $H = p^2 + e^{2ix}$ and the free particle $\mathbb{H} = p^2$ as given by solutions of the entwining equation

$$H(x,p) \star \widetilde{S}(x,p) = \widetilde{S}(x,p) \star p^2$$

For the Liouville $\longleftrightarrow$ free-particle case, this amounts to a first order PDE similar to that for $\widetilde{G}$, but inherently complex.

$$2ip \frac{\partial}{\partial x} \widetilde{S}(x,p) = e^{2ix} \widetilde{S}(x,p-1)$$

Once again solutions are easily found through the use of a product ansatz. For any value of a parameter s, we immediately find two classes of solutions:

$$\widetilde{S}(x,p;s) = \frac{1}{s^p \Gamma(1+p)} \exp\left(-\frac{1}{4} s \exp(2ix)\right)$$

$$\widetilde{S}_{\text{other}}(x,p;s) = \frac{1}{s^p} \Gamma(-p) \exp\left(\frac{1}{4} s \exp(2ix)\right)$$

The first of these is a "good" solution for $p \in (-1,\infty)$, say, while the second is good for $p \in (-\infty,0)$, thereby providing a pair of solutions that cover the entire real p axis, but *not* so easily joined together.

The dual metric as an absolute ★ square

Each such solution for $\widetilde{S}$ leads to a candidate real metric, given by

$$\widetilde{G} \;=\; \widetilde{S} \star \overline{\widetilde{S}}$$

To verify this, we note that the entwining equation for $\widetilde{S}$, and its conjugate $\overline{\widetilde{S}}$,

$$H \star \widetilde{S} \;=\; \widetilde{S} \star p^2 \;, \quad p^2 \star \overline{\widetilde{S}} = \overline{\widetilde{S}} \star \overline{H}$$

may be combined with the associativity of the star product to obtain

$$H \star \widetilde{S} \star \overline{\widetilde{S}} \;=\; \widetilde{S} \star p^2 \star \overline{\widetilde{S}} = \widetilde{S} \star \overline{\widetilde{S}} \star \overline{H}$$

For the first class of $\widetilde{S}$ solutions, by choosing $s = \pm 2$, and again using the standard integral representation for $1/\Gamma$, we find a result that coincides with the previous composite dual metric.

$$\widetilde{S}\left(x,p;\pm 2\right) \star \overline{\widetilde{S}}\left(x,p;\pm 2\right) \;=\; \frac{\left(\sin^2 x\right)^p}{\left(\Gamma\left(p+1\right)\right)^2} = \widetilde{G}\left(x,p\right)$$

This proves the corresponding operator is positive (perhaps positive definite) and provides a greater appreciation of the ★ roots of $\widetilde{G}$.

An aside ... the ix^3 potential

In this case, it is *not quite* so easy to find a useful class of exact solutions to the entwining equation.

$$H = \frac{1}{2}p^2 + ix^3$$

$$H(x,p) \star \widetilde{G}(x,p) = \widetilde{G}(x,p) \star \overline{H(x,p)}$$

$$\left(p\partial_x - 2x^3 + \frac{3}{2}x\partial_p^2\right)\widetilde{G}(x,p) = 0$$

Nevertheless, *a general real solution* to this parabolic equation is given by the following double-integral representation [7].

$$\widetilde{G}(x,p;[F]) = \iint dtdz\ F\left(2iz + t^2\right)\ \exp\left(ix^2t - ipz + \frac{3}{4}itz^2 + \frac{1}{2}t^3z - \frac{1}{10}it^5\right)$$

where F is an arbitrary function (with suitably nice behavior!). This $\widetilde{G}$ is real for real F.

That is to say, *the general solution is actually a generalized Airy transform.*

However, I have *not yet* been able to write linear combinations of these solutions as a "star square" as in the Liouville case. That would be technically sweet, especially insofar as it could a proof the spectrum is real, different from the proof given by Dorey et al. [15].

The solution is straightforward to obtain by Fourier transforms. Including a linear term, $H = \frac{1}{2}p^2 + ix^3 + \frac{1}{2}i\lambda x$, $H \star \widetilde{G} = \widetilde{G} \star \overline{H}$ boils down to

$$\left(-\lambda x + p\partial_x - 2x^3 + \frac{3}{2}x\partial_p^2\right)\widetilde{G} = 0$$

$$\left(-\frac{1}{2}\lambda + p\partial_q - q + \frac{3}{4}\partial_p^2\right)\widetilde{G} = 0 \quad \text{where} \quad q \equiv x^2$$

$$\left(-\frac{1}{2}\lambda + t\partial_z - i\partial_t - \frac{3}{4}z^2\right)\left[G = \int\int \exp\left(-iqt + ipz\right)\widetilde{G}\right] = 0$$

It is easy to determine the general solution of this first-order equation, and then transform back.

$$G = F\left(2iz + t^2\right)\exp\left(-\frac{1}{10}it^5 + \frac{1}{2}t^3z + \frac{3}{4}itz^2 + \frac{1}{2}i\lambda t\right)$$

$$\widetilde{G} = \iint dtdz\ F\left(2iz + t^2\right)\ \exp\left(iqt - ipz + \frac{1}{2}it\lambda + \frac{3}{4}itz^2 + \frac{1}{2}t^3z - \frac{1}{10}it^5\right)$$

$$m\left(x,p,t,z\right) \equiv F\left(2iz + t^2\right)\ \exp\left(ix^2t - ipz + \frac{1}{2}it\lambda + \frac{3}{4}itz^2 + \frac{1}{2}t^3z - \frac{1}{10}it^5\right)$$

$$\left(-\lambda x + p\frac{\partial}{\partial x} - 2x^3 + \frac{3}{2}x\frac{\partial^2}{\partial p^2}\right)m\left(x,p,t,z\right) = -2x\left(t\frac{\partial}{\partial z} - i\frac{\partial}{\partial t}\right)m\left(x,p,t,z\right)$$

$$\widetilde{G}\left(x,p;[F]\right) = \iint dtdz\ F\left(2iz + t^2\right)\ \exp\left(ix^2t - ipz + \frac{1}{2}it\lambda + \frac{3}{4}itz^2 + \frac{1}{2}t^3z - \frac{1}{10}it^5\right)$$

Or, after changing variables, including rescalings,

$$\widetilde{G}\left(x,p;[F]\right) \;=\; \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} du\; F\left(u\right)\; \exp\left(\frac{1}{2}\left(t^2-u\right)p\right)\; \exp\left(ix^2t-\frac{1}{4}iu^2t+\frac{1}{6}iut^3-\frac{1}{20}it^5\right)$$

I leave it as an exercise to choose specific functions to reduce this to a more manageable form. For example, when F is a Gaussian with parameters a and b, we obtain an ugly result that only a mother could love.

$$\widetilde{G}\left(x,p;\left[F\left(u\right)=\frac{1}{\sqrt{\pi}}\exp\left(-a\left(u-b\right)^2\right)\right]\right)=$$

$$e^{-ab^2}\int_{-\infty}^{+\infty} dt\; \frac{1}{\sqrt{a+\frac{1}{4}it}}\; \exp\left(\frac{\frac{1}{16}\left(p-4ab\right)^2+ix^2at+\left(\frac{1}{2}ap-\frac{1}{4}x^2\right)t^2+\frac{1}{12}i\left(p+2ab\right)t^3-\frac{1}{20}iat^5+\frac{1}{180}t^6}{a+\frac{1}{4}it}\right)$$

Still, for $a>0$ this result has the virtue that the final t integral is very nicely convergent for all real x and p, and numerical evaluation of this function is *not* difficult.

$$\widetilde{G}\left(x=1,p=1;\left[F\left(u\right)=\frac{1}{\sqrt{\pi}}\exp\left(-u^2\right)\right]\right) \;=\; +7.8163$$

$$\widetilde{G}\left(x=3,p=1;\left[F\left(u\right)=\frac{1}{\sqrt{\pi}}\exp\left(-u^2\right)\right]\right) \;=\; -2.2335$$

etc.

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{a+\frac{1}{4}it}}\exp\left(\frac{\frac{1}{16}(p-4ab)^2+ix^2at+\left(\frac{1}{2}ap-\frac{1}{4}x^2\right)t^2+\frac{1}{12}i(p+2ab)t^3-\frac{1}{20}iat^5+\frac{1}{180}t^6}{a+\frac{1}{4}it}\right)dt\Big|_{a=1,b=0,p=1,x=3}$$
$$=-2.2335-1.0351\times10^{-28}i$$

What has the Liouville metric got to do with wave functions?

The eigenvalue problem is well-posed if wave functions are required to be bounded (free particle bc's).

$$\left(-\frac{\partial^2}{\partial x^2} + m^2 e^{2ix}\right)\psi_E \quad = \quad E\psi_E$$

We find all real $E \geq 0$ are allowed.

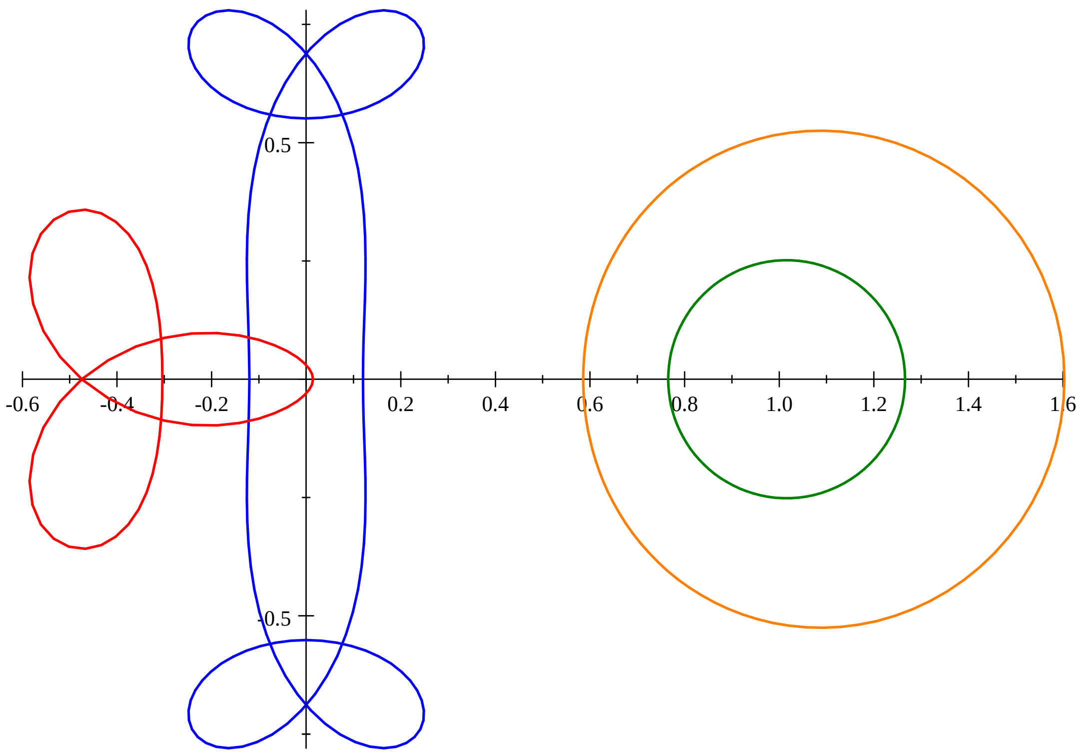
$$z = me^{ix}$$

gives Bessel's equation.

$$J_{\pm\sqrt{E}}\left(me^{ix}\right) \quad = \quad \left(\frac{m}{2}e^{ix}\right)^{\pm\sqrt{E}} \sum_{n=0}^{\infty} \frac{\left(-m^2/4\right)^n}{n!\Gamma\left(1+n\pm\sqrt{E}\right)} e^{2inx}$$

An $E = 0$ solution exists!

Representative wave functions.



$\left(\mathrm{Re}\,\psi_E(x), \mathrm{Im}\,\psi_E\right)$ and $\left(\mathrm{Re}\,\psi_E^2, \mathrm{Im}\,\psi_E^2\right)$, for $E = 0$ (green, orange) and for $E = 1/4$ (blue, red)

Integral representations for $E = n^2$ and quantum equivalence to a free particle on a circle

The 2π-periodic Bessel functions are in fact the canonical integral transforms of free plane waves on a circle, as constructed in this special situation just by exponentiating the classical generating function. Explicitly,

$$J_n\left(me^{ix}\right) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(-in\theta\right)\exp\left(ime^{ix}\sin\theta\right)d\theta\ , \quad n \in \mathbb{Z}$$

with $J_{-n}(z) = (-)^n J_n(z)$. The integral transform is a two-to-one map from the space of all free particle plane waves to Bessel functions ($e^{\mp in\theta} \to (\pm 1)^n J_n$). But acting on the linear combinations $e^{in\theta} + (-)^n e^{-in\theta}$ the kernel gives a map which is one-to-one, hence invertible on this subspace. The situation here is exactly like the real Liouville QM, for all positive energies, except for the fact that *here we have a well-behaved ground state.*

While this is a well-known integral representation of the Bessel function, appearing in hundreds of books, an interpretation as *a canonical transformation* is universally overlooked as far as I can tell.

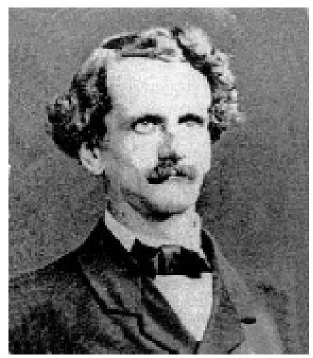
What are the dual wave functions?

The "PT method" of constructing the dual space by simply changing normalizations and phases of the wave functions does *not* provide a biorthonormalizable set of functions in this case, since

$$\frac{1}{2\pi}\int_0^{2\pi} J_k\left(me^{ix}\right) J_n\left(me^{ix}\right) dx \;=\; \begin{cases} 1 & \text{if} \quad k=n=0 \\ 0 & \quad\text{otherwise} \end{cases}$$

This follows because the Js are series in only positive powers of e^{ix}. So all the 2π-periodic energy eigenfunctions are *self-orthogonal* except for the ground state.

In retrospect, this difficulty was circumvented by Carl in the mid-19th century (that's Carl Neumann ... not Carl Bender).



Carl Neumann was Bessel's nephew ... his sister-in-law's son, actually.

A simple 2π-periodic biorthogonal system

Elements of the dual space for the 2π-periodic eigenfunctions are given by Neumann polynomials, $\{A_n\}$. For all analytic Bessel functions of non-negative integer index

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k}$$

there are corresponding associated *Neumann polynomials* in powers of $1/z$ that are dual to $\{J_n\}$ on any contour enclosing the origin. These are given by

$$A_0(z) = 1\,, \quad A_1(z) = \frac{2}{z}\,, \quad A_{n\geq 2}(z) = n\left(\frac{2}{z}\right)^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k}$$

These satisfy an *inhomogeneous* equation (cf. the 2×2 matrix example from Kato's book, mentioned previously, especially at exceptional spectral points) where the inhomogeneity is orthogonal to all the $J_k(z)$.

$$\begin{aligned}
-\frac{d^2}{dx^2} A_n\left(me^{ix}\right) + \left(m^2 e^{2ix} - n^2\right) A_n\left(me^{ix}\right) &= \begin{cases} 2nme^{ix} & \text{for odd} \quad n \\ 2m^2 e^{2ix} & \text{for even} \quad n \neq 0 \end{cases} \\
-\frac{d^2}{dx^2} J_n\left(me^{ix}\right) + \left(m^2 e^{2ix} - n^2\right) J_n\left(me^{ix}\right) &= 0
\end{aligned}$$

The inhomogeneities[1] here are actually linear combinations of an *infinite* number of $J_k\left(me^{ix}\right)$.

[1]Note: The inhomogeneity is $m^2 e^{2ix}$ for the special case $n = 0$.

Re-expressed for the imaginary Liouville problem, the key orthogonality relation is now

$$\frac{1}{2\pi}\int_0^{2\pi} A_k\left(me^{ix}\right) J_n\left(me^{ix}\right) dx \quad = \quad \delta_{kn}$$

Hence the metric on the space of dual wave functions (i.e. G^{-1}) is

$$J\left(x,y\right) \quad \equiv \quad J_0\left(me^{-ix} - me^{iy}\right) = \sum_{n=0}^{\infty} \varepsilon_n J_n\left(me^{-ix}\right) J_n\left(me^{iy}\right)$$

where $\varepsilon_0 = 1,\ \varepsilon_{n\neq 0} = 2.$

This manifestly hermitian, bilocal kernel $J(x,y) = J(y,x)^*$ can be used to evaluate the norm of a general function in the span of the eigenfunctions

$$\psi(x) \equiv \sum_{n=0}^{\infty} c_n \sqrt{\varepsilon_n} J_n\left(me^{ix}\right)$$

through use of the corresponding dual function

$$\psi_{\text{dual}}(x) \equiv \sum_{n=0}^{\infty} c_n^* A_n\left(me^{ix}\right) / \sqrt{\varepsilon_n}$$

where once again $\varepsilon_0 = 1$, $\varepsilon_{n\neq 0} = 2$. The result is as expected.

$$\|\psi\|^2 = \frac{1}{(2\pi)^2} \int_0^{2\pi} dx \int_0^{2\pi} dy \, \overline{\psi_{\text{dual}}(x)} \; J(x,y) \; \psi_{\text{dual}}(y) = \sum_{n=0}^{\infty} |c_n|^2$$

Wigner transform of the bilocal metric

A scalar product for a biorthogonal system such as $\{A_k, J_n\}$ can always be written as an integral over a doubled configuration space involving a "bilocal metric" $J(x,y)$.

$$(\phi, \psi) = \iint \phi(x)\, J(x,y)\, \psi(y)\, dxdy$$

Bilocal $\leftrightarrow$ phase space When a scalar product is so expressed it is easily re-expressed in phase space (which we suppose to be $\mathbb{R}^2$ in this paragraph) through the use of a Wigner transform.

$$f_{\psi\phi}(x,p) \equiv \frac{1}{2\pi} \int e^{iyp}\ \psi\left(x - \frac{1}{2}y\right) \phi\left(x + \frac{1}{2}y\right) dy$$

Fourier inverting gives the point-split product

$$\phi(x)\, \psi(y) = \int_{-\infty}^{\infty} e^{i(y-x)p}\ f_{\psi\phi}\left(\frac{x+y}{2}, p\right) dp$$

Thus the scalar product can be re-written as

$$(\phi, \psi) = \iint G(x,p)\ \ f_{\psi\phi}(x,p)\, dxdp$$

where the phase-space metric is the Wigner transform of the bilocal metric.

$$G(x,p) = \int e^{iyp}\ J\left(x - \frac{1}{2}y, x + \frac{1}{2}y\right) dy$$

and inversely

$$J(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-y)p}\ G\left(\frac{x+y}{2}, p\right) dp$$

In a more abstract notation the form of the scalar product is $\left(\overline{\phi}, \psi\right) = Tr\left(\mathsf{G}\left|\psi\right\rangle\left\langle\phi\right|\right) = Tr\left(\left|\psi\right\rangle \widetilde{\left\langle\phi\right|}\right)$.

Liouville dual metric

To be more specific, for 2π-periodic *dual* functions of imaginary Liouville quantum mechanics, the scalar product given previously can be re-expressed in a form which is immediately converted to phase-space.

$$J\left(x,y\right) = J_0\left(-2ie^{i(y-x)/2}\sin\left(\frac{x+y}{2}\right)\right)$$

Up to some normalization (and other) conventions, the corresponding metric in phase space is given by the Wigner transform of this bilocal. Namely

$$\widetilde{G}\left(x,p\right) = \frac{1}{2\pi}\int_0^{2\pi} J\left(x+w,x-w\right)e^{2iwp}dw = \frac{1}{2\pi}\int_0^{2\pi} J_0\left(-2ie^{-iw}\sin x\right)e^{2iwp}dw$$

Hence the simple final answer.

$$\widetilde{G}\left(x,p\right) = \frac{\left(\sin^2 x\right)^p}{\left(p!\right)^2} \quad \text{for integer } p \geq 0\text{, but vanishes for integer } p<0$$

Thus we re-obtain the previous solution of the entwining equation.

An equivalent operator expression can be obtained by the method of Weyl transforms. Rather than pursue this, let us move on to the field theory extension of the model.

Liouville field theory functionals

Consider the Liouville field theory on a spacetime cylinder, with fields periodic in the spatial variable σ. The conventional energy density in the Schrödinger functional formalism is

$$\mathcal{H}_\phi(\sigma) = -\frac{\delta^2}{\delta\phi(\sigma)^2} + (\partial_\sigma \phi(\sigma))^2 + e^{2i\phi(\sigma)} \tag{1}$$

I've written this for the potential $\exp(2b\phi)$ when $b = i$. I leave it as an exercise to work out the results that follow for other $b = i\beta$. But note [10] there is an instability (phase transformation?) when $\beta^2 > 2\pi$.

Classical Liouville and free field configurations are related by the local version of the classical point particle canonical transformation, with generator $F = me^{ix}\sin\theta$, only augmented with a duality term.

$$\begin{aligned} F[\phi,\psi] &= \int_\sigma \phi(\sigma)\,\partial_\sigma\psi(\sigma) + me^{i\phi(\sigma)}\sin\psi(\sigma) \\ \pi(\sigma) &= \frac{\delta}{\delta\phi(\sigma)}F[\phi,\psi] = \partial_\sigma\psi(\sigma) + ime^{i\phi(\sigma)}\sin\psi(\sigma) \\ \varpi(\sigma) &= -\frac{\delta}{\delta\psi(\sigma)}F[\phi,\psi] = \partial_\sigma\phi(\sigma) - me^{i\phi(\sigma)}\cos\psi(\sigma) \end{aligned} \tag{2}$$

The single-particle result expressing periodic Liouville QM wave functions as transforms of free particle wave functions for integer $\sqrt{E}$ also has an immediate generalization in terms of Schrödinger wave functionals. Liouville and free field energy eigenfunctionals are related, without any approximations, by a functional transform (suitably regularized) that involves just the classical generating functional connecting the two fields, if we suppose at least the zero mode of the free field is periodic.

$$\Phi_E\left[\phi\right] \;=\; \int d\mu\left[\psi\right]\; e^{iF[\phi,\psi]}\;\Psi_E\left[\psi\right] \tag{3}$$

where $d\mu\left[\psi\right] = \Pi_\sigma d\psi\left(\sigma\right) = \Pi_n d\psi_n$ is the usual functional measure, and where $\Psi_E\left[\psi\right]$ is a free massless field energy eigenfunctional. We will use the prescription that the zero mode ψ_0 is integrated over $[0, 2\pi]$ while the non-zero modes $\psi_{n\neq 0}$ are integrated over $(-\infty, +\infty)$, with $\Psi_E\left[\psi\right]$ given by plane-waves in the zero mode, and a Gaussian in the non-zero modes.

For example, the vacuum functional is $\Psi_0\left[\psi\right] = \exp\left(-\frac{1}{2}\int_\sigma\ \psi\left(\sigma\right)\left|\partial_\sigma\right|\psi\left(\sigma\right)\right)$ with $\left|\partial_\sigma\right| \equiv \sqrt{-\partial_\sigma^2}$. Translationally invariant, higher energy configurations follow from simple boosts that differ from $\Psi_0\left[\psi\right]$ only through a zero mode plane wave: $\Psi_0\left[\psi\right] \rightarrow \Psi_k\left[\psi\right] = \Psi_0\left[\psi\right]\exp\left(ik\int_\sigma\ \psi\left(\sigma\right)\right)$.

To confirm this connection between energy functionals, we define

$$D_{\psi}(\sigma) = -i\frac{\delta}{\delta\psi(\sigma)} + \partial_{\sigma}\phi(\sigma) \ , \quad D_{\phi}(\sigma) = -i\frac{\delta}{\delta\phi(\sigma)} - \partial_{\sigma}\psi(\sigma) \tag{4}$$

observe that $[D_{\psi}(\sigma_1), D_{\phi}(\sigma_2)] = 0 = \left[D_{\psi}(\sigma_1) \pm D_{\phi}(\sigma_1), e^{i\phi(\sigma_2)\mp i\psi(\sigma_2)}\right]$, even for $\sigma_1 = \sigma_2$, and compute

$$\begin{aligned} (D_{\psi}(\sigma) \pm D_{\phi}(\sigma))\ e^{iF} &= me^{i\phi(\sigma)\pm i\psi(\sigma)}\ e^{iF} \\ (D_{\psi}(\sigma_1) + D_{\phi}(\sigma_1))(D_{\psi}(\sigma_2) - D_{\phi}(\sigma_2))\ e^{iF} &= e^{i\phi(\sigma_2)+i\phi(\sigma_1)+i\psi(\sigma_1)-i\psi(\sigma_2)}\ e^{iF} \end{aligned} \tag{5}$$

Taking $\sigma_1 = \sigma_2$ does not produce a singularity. So we have the *exact* result

$$\left(D_{\psi}^2(\sigma) - D_{\phi}^2(\sigma)\right)\ e^{iF} = e^{2i\phi(\sigma)}\ e^{iF} \tag{6}$$

On the other hand

$$\begin{gathered} \left(D_{\psi}^2(\sigma) - D_{\phi}^2(\sigma)\right)\ e^{iF} \equiv \\ \left(-\frac{\delta^2}{\delta\psi(\sigma)^2} + (\partial_{\sigma}\psi(\sigma))^2 + \frac{\delta^2}{\delta\phi(\sigma)^2} - (\partial_{\sigma}\phi(\sigma))^2 - 2i\partial_{\sigma}\left(e^{i\phi(\sigma)}\cos\psi(\sigma)\right)\right)\ e^{iF} \end{gathered} \tag{7}$$

So, expressed in terms of the (un-improved) energy density operators for Liouville and free fields, in the Schrödinger wave functional formalism,

$$\mathcal{H}_{\psi}(\sigma) = -\frac{\delta^2}{\delta\psi(\sigma)^2} + (\partial_\sigma \psi(\sigma))^2 \tag{8}$$

$$\mathcal{H}_{\phi}(\sigma) = -\frac{\delta^2}{\delta\phi(\sigma)^2} + (\partial_\sigma \phi(\sigma))^2 + e^{2i\phi(\sigma)} \tag{9}$$

we are led to the exact relation

$$\mathcal{H}_{\psi}(\sigma)\ e^{iF} = \mathcal{H}_{\phi}(\sigma)\ e^{iF} + 2i\partial_\sigma \left(e^{i\phi(\sigma)} \cos\psi(\sigma)\right)\ e^{iF} \tag{10}$$

Clearly the total space derivative on the RHS arises from the conformal improvements for the Liouville and free field energy densities. It will drop out of the total energy, if periodic boundary conditions are assumed, but not the Virasoro charges.

Thus the imaginary Liouville energy density, and indeed the entire local conformally improved energy-momentum density tensor, entwines to that of a free field. Hence the functional transform given above yields exact results for Liouville energy eigenfunctionals.

$$\mathcal{H}_\phi(\sigma)\,\Phi_E[\phi] = \int d\mu[\psi]\ \left(\mathcal{H}_\psi(\sigma) - 2i\partial_\sigma\left(e^{i\phi(\sigma)}\cos\psi(\sigma)\right)\right) e^{iF[\phi,\psi]}\ \Psi_E[\psi] \tag{11}$$

$$H_\phi = \int_\sigma \mathcal{H}_\phi(\sigma)\ ,\quad H_\psi = \int_\sigma \mathcal{H}_\psi(\sigma) \tag{12}$$

$$H_\phi\Phi_E[\phi] = \int d\mu[\psi]\ H_\psi e^{iF[\phi,\psi]}\ \Psi_E[\psi] = \int d\mu[\psi]\ e^{iF[\phi,\psi]}\ H_\psi\Psi_E[\psi] = E\Phi_E[\phi] \tag{13}$$

where in the second step of the last equation, we have functionally integrated by parts and assumed no end-point contributions[2]. For instance, if the range of integration is $[0, 2\pi]$ for the zero mode, periodicity of each factor in the integrand, including $\Psi_E[\psi]$, ensures there are no zero-mode end-point contributions, while if the range is $(-\infty, \infty)$ for each non-zero mode, the Gaussian behavior of $\Psi_E[\psi]$ ensures there are no end-point effects for the non-zero modes.

[2] All this looks much cleaner in a functional phase-space formalism, albeit somewhat heuristic. We leave this approach as an exercise for the interested student.

The story for the dual wave functionals is almost the same, except that the functional integration now has a lower limit of integration (namely $\psi = 0$) which contributes end-point terms under integrations by parts. Modulo normalizations and an additive constant, we have

$$\Phi_E^{\text{dual}}[\phi] \;=\; \int_{\psi=0}^{i\infty} d\mu\,[\psi]\;\; e^{iF[\phi,\psi]}\;\; \Psi_E\,[\psi] \tag{14}$$

Note that we do not distinguish the dual free field eigenfunctional $\Psi_E^*[\psi]$ from a generic $\Psi_E[\psi]$ (although for clarity, perhaps we should.) The upper limit "$i\infty$" is somewhat ambiguous. The idea is just to choose the upper limit for each mode to be a point in the complex plane such that there is no end-point contribution upon integration by parts. In general this point is reached by following a contour that depends discontinuously on the phase of the ϕ modes, i.e. the upper limit is different for a discrete number of "phase wedges" of ϕ. For example, if we consider only the zero modes, with ϕ_0 real, the upper limit for the ψ_0 integration is actually $i\infty - \alpha$ where $|\alpha + \phi_0| < \pi/2$, thereby ensuring that $\cos(\alpha + \phi_0) > 0$ and damping the integrand. These discontinuous contour changes require some careful attention in calculations.

In any case, we now claim the dual functional satisfies an *inhomogeneous* equation.

$$\begin{aligned}\mathcal{H}_{\phi}(\sigma)\,\Phi_E^{\text{dual}}[\phi] &= \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; \left(\mathcal{H}_{\psi}(\sigma) - 2i\partial_{\sigma}\left(e^{i\phi(\sigma)}\cos\psi(\sigma)\right)\right)\Psi_E[\psi] \\ &\quad -i\left(D_{\psi}(\sigma) - me^{i\phi(\sigma)}\right)\Psi_E[\psi]\Big|_{\psi=0} \end{aligned} \tag{15}$$

Hence

$$\begin{aligned}H_{\phi}\Phi_E^{\text{dual}}[\phi] &= \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; H_{\psi}\Psi_E[\psi] \\ &\quad -i\int_{\sigma}\;\left(D_{\psi}(\sigma) - me^{i\phi(\sigma)}\right)\Psi_E[\psi]\Big|_{\psi=0} \end{aligned} \tag{16}$$

where $\int_{\sigma}\;\left(D_{\psi}(\sigma) - me^{i\phi(\sigma)}\right) = \int_{\sigma}\;\left(-i\frac{\delta}{\delta\psi(\sigma)} - me^{i\phi(\sigma)}\right)$. This is the first inhomogeneous functional equation that I have considered.

Some details. First

$$\begin{aligned}
\mp D_\phi(\sigma)\,\Phi_E^{\text{dual}}[\phi] &= \mp\int_{\psi=0}^{i\infty} d\mu[\psi]\;\; D_\phi(\sigma)\, e^{iF[\phi,\psi]}\;\; \Psi_E[\psi] = \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; \left(D_\psi(\sigma) - me^{i\phi(\sigma)\pm i\psi(\sigma)}\right) e^{iF[\phi,\psi]}\;\; \Psi_E[\psi] \\
&= ie^{iF[\phi,\psi=0]}\;\; \Psi_E[\psi=0] + \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; \left(D_\psi(\sigma) - me^{i\phi(\sigma)\pm i\psi(\sigma)}\right)\Psi_E[\psi] \\
&= i\Psi_E[\psi=0] + \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; \left(D_\psi(\sigma) - me^{i\phi(\sigma)\pm i\psi(\sigma)}\right)\Psi_E[\psi] \qquad (17)
\end{aligned}$$

Note the sign flips in the exponentials $e^{i\phi(\sigma)\pm i\psi(\sigma)}$. Then there is

$$\left(\mathcal{H}_\phi(\sigma) + 2i\partial_\sigma\left(e^{i\phi(\sigma)}\cos\psi(\sigma)\right)\right)\Phi_E^{\text{dual}}[\phi]$$

$$= \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; \left(\mathcal{H}_\phi(\sigma) + 2i\partial_\sigma\left(e^{i\phi(\sigma)}\cos\psi(\sigma)\right)\right) e^{iF[\phi,\psi]}\;\; \Psi_E[\psi] = \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; \mathcal{H}_\psi(\sigma)\, e^{iF[\phi,\psi]}\;\; \Psi_E[\psi]$$

$$= \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; \left(-\frac{\delta^2}{\delta\psi(\sigma)^2} + (\partial_\sigma\psi(\sigma))^2\right) e^{iF[\phi,\psi]}\;\; \Psi_E[\psi]$$

$$= \frac{\delta}{\delta\psi(\sigma)} e^{iF[\phi,\psi]}\bigg|_{\psi=0} \Psi_E[\psi=0] + \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; \frac{\delta}{\delta\psi(\sigma)} e^{iF[\phi,\psi]}\;\; \frac{\delta}{\delta\psi(\sigma)}\Psi_E[\psi] + \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; (\partial_\sigma\psi(\sigma))^2\,\Psi_E[\psi]$$

$$= \frac{\delta}{\delta\psi(\sigma)} e^{iF[\phi,\psi]}\bigg|_{\psi=0} \Psi_E[\psi=0] - e^{iF[\phi,\psi=0]}\;\; \frac{\delta}{\delta\psi(\sigma)}\Psi_E[\psi]\bigg|_{\psi=0} + \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; \mathcal{H}_\psi(\sigma)\,\Psi_E[\psi]$$

$$= -i\left[\left(D_\psi(\sigma) - me^{i\phi(\sigma)}\right)\Psi_E[\psi]\big|_{\psi=0} + \int_{\psi=0}^{i\infty} d\mu[\psi]\;\; e^{iF[\phi,\psi]}\;\; \mathcal{H}_\psi(\sigma)\,\Psi_E[\psi]\right.$$

since $e^{iF[\phi,\psi=0]} = 1$ and $\frac{\delta}{\delta\psi(\sigma)} e^{iF[\phi,\psi]}\Big|_{\psi=0} = ime^{i\phi(\sigma)} - i\partial_\sigma\phi(\sigma)$. This is the result advertised earlier.

For non-periodic zero modes, there is also a functional generalization of the Schläfli and Sonine representation.

$$\begin{aligned} J_{\pm\sqrt{E}}\left(me^{ix}\right) &= \left(\frac{m}{2}e^{ix}\right)^{\pm\sqrt{E}} \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{dw}{w}\, w^{\mp\sqrt{E}} \exp\left(w - \frac{m^2 e^{2ix}}{4w}\right) \\ &= \frac{1}{2\pi i}\int_{-\infty\times\frac{2}{m}e^{-ix}}^{(0+)} \frac{du}{u}\, u^{\mp\sqrt{E}} \exp\left(\frac{1}{2}me^{ix}\left(u - \frac{1}{u}\right)\right) \end{aligned} \tag{18}$$

Up to normalizations, and other things, it looks just like the previous (3).

$$\Phi_E\left[\phi\right] = \int_{\mathcal{C}} d\mu\left[\psi\right]\ e^{iF[\phi,\psi]}\ \Psi_E\left[\psi\right] \tag{19}$$

with F as given previously. But the contour $\mathcal{C}$ requires some discussion. For the zero mode ψ_0 (or rather, an appropriate logarithm of ψ_0) it is just the contour in Sonine's integral representation for the Γ function. For the non-zero modes, something similar is required. But since we will expand $e^{iF[\phi,\psi]}$ in the non-zero modes, and since $\Psi_E\left[\psi\right]$ is a Gaussian in them, it suffices just to integrate the non-zero modes along the real axis. That is to say, for each term in the perturbation series as we will define it, the non-zero mode contour can always be deformed to the real axis.

The message to be taken from all this is imaginary Liouville field theory is an interesting example of a Q-hermitian theory, with a real energy spectrum, and it is indeed a consistent quantum field theory. Still, it remains to compute expectations of various Liouville fields using these integral representations. This is in progress.

Things to do:

- Show that imaginary Liouville perturbation theory agrees with Schomerus' 3-point function. This would place the latter on a firmer foundation, in my opinion.[3]

- Better yet, show that the representation of the Liouville energy eigenstates as functional integral transforms of free field states leads exactly to the cited 3-point function.

- Connect the exact functional expressions to exact path integral results for the imaginary Liouville Q-hermitian Hamiltonian.

- Etc. for fields on supermanifolds.

[3] The matrix element of a product of $n-2$ exponentials on the cylinder is related to the correlator of n exponentials on a sphere by associating the states with two points on the sphere represented on the complex plane by, say $x_1 = 0$ and $x_n = \infty$. Putting $\alpha_1 = Q/2+iP$ and $\alpha_n = Q/2+iP'$, with P, P' real, the energies of the two states participating in the matrix element are given by $E = 2P^2$ and $E' = 2P'^2$. Then, in perturbation theory, we would need to compute the matrix elements of the operator $e^{2i\beta\lambda\phi}$ between states with $P = \beta k$ and $P' = \beta k'$, and compare them with the asymptotic series of Schomerus' 3-point function, in the limit $\beta \to 0$. For real Liouville theory, Charles Thorn showed these perturbative results were consistent with the exact 3-point function of Dorn and Otto, and the Zamolodchikovs. It would be good to do the analogous demonstration for imaginary Liouville theory.

Conclusions:

Is there a quasi-hermitian Hamiltonian in your future ?

You can be pseudo-certain of it !

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1 Appendix: Classical Liouville dynamics

For the imaginary Liouville theory

$$H\left(x,p\right)=p^{2}+\exp\left(2ix\right)$$

the classical story goes as follows, if we follow Xavier and de Aguiar [29], and Bender [4] et al. Even for complex x and p we solve

$$\frac{dx\left(t\right)}{dt}=\frac{\partial H}{\partial p}=2p\left(t\right)\ ,\quad \frac{dp\left(t\right)}{dt}=-\frac{\partial H}{\partial x}=-2im^{2}e^{2ix(t)}$$

i.e. we postulate Poisson brackets such that $\{x,p\}=1$, even on $\mathbb{C}^2$. Complex energy conservation allows reduction to a single first order equation

$$\frac{dx\left(t\right)}{dt}=2p\left(t\right)=\pm2\sqrt{E-m^{2}e^{2ix(t)}}$$

For $E\neq0$,

$$x\left(t\right)=\frac{1}{2i}\ln\left(\frac{E/m^{2}}{\cosh^{2}\left(\mp2i\sqrt{E}t+\operatorname{arctanh}\sqrt{1-\frac{m^{2}}{E}\exp\left(2ix\left(0\right)\right)}\right)}\right)$$

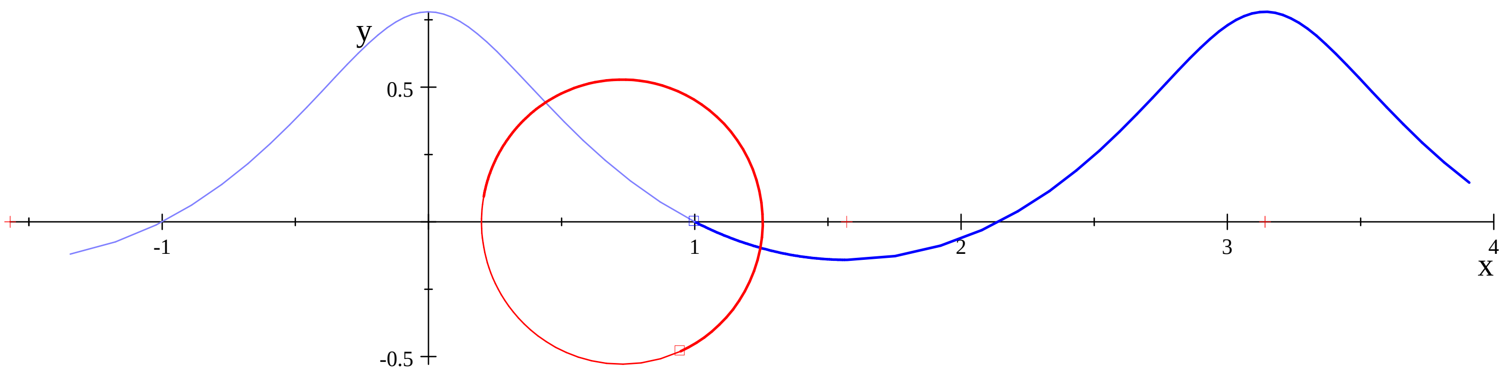
which is exact, but not very transparent.

On the other hand,

$$p(t) = \pm\sqrt{E}\left(\frac{a(E,x_0) - \exp\left(\pm 4i\sqrt{E}t\right)}{a(E,x_0) + \exp\left(\pm 4i\sqrt{E}t\right)}\right)$$

is somewhat easier to understand. For real E this is a circle in the complex plane, of radius $\frac{2|a|}{|a|^2-1}\sqrt{E}$, whose center is on the real axis at $\pm\frac{|a|^2+1}{|a|^2-1}\sqrt{E}$.

Trajectories look like this.

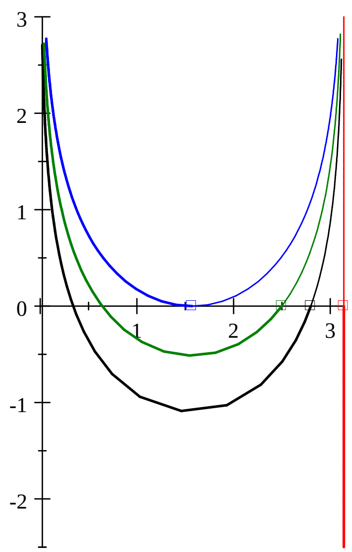


$(\operatorname{Re} x, \operatorname{Im} x)$ in blue and $(\operatorname{Re} p, \operatorname{Im} p)$ in red, plotted parametrically for $E = 1/4$

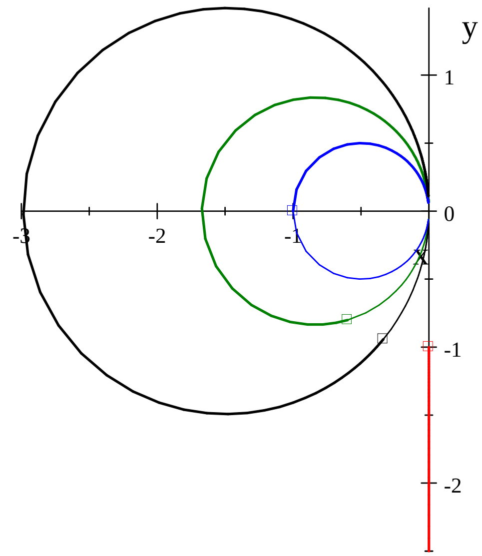
The $E=0$ motion is simpler, but still has interesting structure, as follows.

$$x_{E=0}(t) = i \ln\left(e^{-ix(0)} \pm 2mt\right) \ , \quad \frac{dx_{E=0}(t)}{dt} = \frac{\pm 2im}{e^{-ix(0)} \pm 2mt}$$

For various initial x and p, we plot the trajectories parametrically.



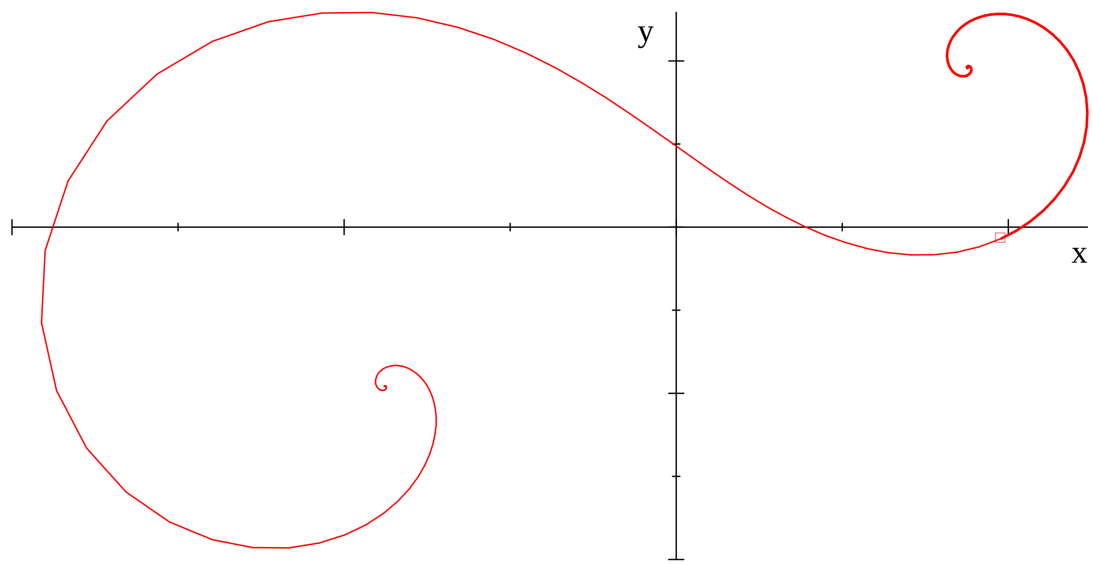
$(\mathrm{Re}\,x, \mathrm{Im}\,x)$ for $E=0$.



$(\mathrm{Re}\,p, \mathrm{Im}\,p)$ for $E=0$.

The vertical red line is $\mathrm{Re}\,x = \pi$.

For artistic purposes, here is a momentum trajectory for complex $E = e^{i}$ and for $x(0) = 1$. The flow is between symmetrical complex fixed points at $p_{\pm} = \pm e^{i/2} = \pm 0.877\,58 \pm 0.479\,43\ i$.



This raises the question: What did Stradivarius know and when did he know it?

Canonical transformations

Many of the classical properties are more easily understood upon taking note of the following fact. There exist generating functions for canonical transformations from Liouville to free particle dynamics, with free variables θ and p_θ. However, in the present context, *both* free particle and Liouville dynamics must be complexified, in general. One such transformation is given by

$$F = me^{ix}\sin\theta \ , \quad p \equiv \frac{\partial}{\partial x}F = ime^{ix}\sin\theta \ , \quad p_\theta \equiv -\frac{\partial}{\partial\theta}F = -me^{ix}\cos\theta$$

The free particle's momentum is conserved, $p_\theta = \pm\sqrt{E}$, since under the transformation

$$H \quad = \quad p^2 + m^2 e^{2ix} = p_\theta^2 = H_{\text{free}}$$

Turning points are encountered for real $E > 0$ if and only if $\operatorname{Im}\theta = 0$ on the corresponding free particle trajectories.

2 Appendix: QM in Phase Space – a tutorial

There are three mathematically equivalent but *autonomous formulations of quantum mechanics* based on:

1. states and operators (Hilbert space),
2. path integrals,

& 3. phase-space.




A lot of “old” money was made based on the first two formulations ...

but the third formulation has only received attention relatively recently under the heading of “deformation quantization” even though it has been around longer than path integrals.

Wigner functions (WFs)

WFs are the principal players in the last of these three arenas, and while they are ordinary functions of their c-number arguments, nevertheless, *non-commutativity is* still an *unavoidable* rule of the game.

On the Quantum Correction For Thermodynamic Equilibrium

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Phys. Rev. 40, 749–759 (1932)
Received 14 March 1932

The probability of a configuration is given in classical theory by the Boltzmann formula exp[-V/hT] where V is the potential energy of this configuration. For high temperatures this of course also holds in quantum theory. For lower temperatures, however, a correction term has to be introduced, which can be developed into a power series of h. The formula is developed for this correction by means of a probability function and the result discussed.

"If a wave function $\psi\left(x_1 \cdots x_n\right)$ is given one may build the following expression[§]

$$P\left(x_1, \cdots, x_n ; p_1, \cdots, p_n\right)=\left(\frac{1}{h \pi}\right)^n \int_{-\infty}^{\infty} \cdots \int d y_1 \cdots d y_n \psi\left(x_1+y_1 \cdots x_n+y_n\right)^* \psi\left(x_1-y_1 \cdots x_n-y_n\right) e^{2 i\left(p_1 y_1+\cdots+p_n y_n\right) / h} \qquad (5)$$

and call it the probability-function of the simultaneous values of $x_1 \cdots x_n$ for the coordinates and $p_1 \cdots p_n$ for the momenta. In (5), as throughout this paper, h is the Planck constant divided by 2π and the integration with respect to the y has to be carried out from $-\infty$ to ∞. Expression (5) is real, but not everywhere positive. ...

[§] This expression was found by L. Szilard and the present author some years ago for another purpose."

Wigner functions = Weyl-correspondents of density operators

$$\widehat{\boldsymbol{\rho}}\left(\widehat{\mathbf{x}},\widehat{\mathbf{p}}\right)=\frac{1}{\left(2\pi\hbar\right)^{n}}\int d^{n}Xd^{n}P\int d^{n}xd^{n}p\,f\left(x,p\right)\,\exp\left(iX\cdot\left(\widehat{\mathbf{p}}-p\right)/\hbar+iP\cdot\left(\widehat{\mathbf{x}}-x\right)/\hbar\right)$$

So *WFs* reside in *phase-space*. In one x and one p dimension

$$f\left(x,p\right)=\frac{1}{\pi\hbar}\int dy\ \left\langle x+y\right|\,\widehat{\boldsymbol{\rho}}\,\left|x-y\right\rangle\ e^{-2ipy/\hbar}$$

$$\left\langle x+y\right|\,\widehat{\boldsymbol{\rho}}\,\left|x-y\right\rangle=\int dp\,f\left(x,p\right)\ e^{2ipy/\hbar}$$

$$\widehat{\boldsymbol{\rho}}=2\int dxdy\int dp\ \left|x+y\right\rangle\ f\left(x,p\right)\ e^{2ipy/\hbar}\ \left\langle x-y\right|$$

For a pure-state

$$f\left(x,p\right)=\frac{1}{\pi\hbar}\int dy\,\psi\left(x+y\right)\ \psi^{*}\left(x-y\right)\ e^{-2ipy/\hbar}$$

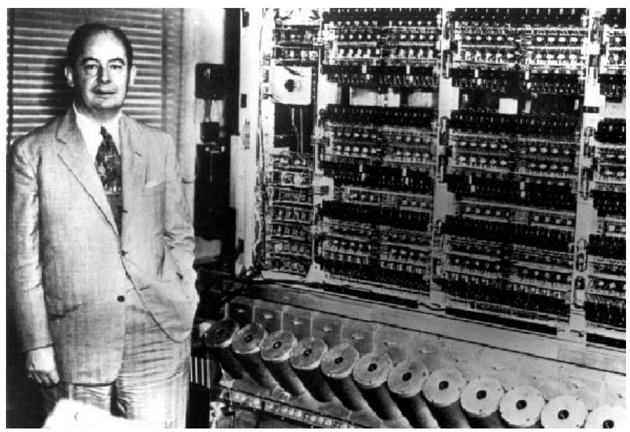
$$\psi\left(x+y\right)\ \psi^{*}\left(x-y\right)=\int dp\,f\left(x,p\right)\ e^{2ipy/\hbar}$$

$$\widehat{\boldsymbol{\rho}}=\left|\psi\right\rangle\left\langle\psi\right|$$

where as usual $\psi\left(x+y\right)=\left\langle x+y\,\middle|\psi\right\rangle,\ \left\langle\psi\right|\,x-y\rangle=\psi^{*}\left(x-y\right).$

More generally, to represent *all* operators on phase-space in a selected basis we need the Weyl-correspondents of arbitrary $|\psi_2\rangle \langle\psi_1|$. We will call these *non-diagonal WFs*

$$f_{12}(x,p) = \frac{1}{\pi\hbar} \int dy\, \psi_1(x+y)\ \psi_2^*(x-y)\ e^{-2ipy/\hbar} = f_{21}^*(x,p)$$



Star products

What about non-commutativity? QM requires it!

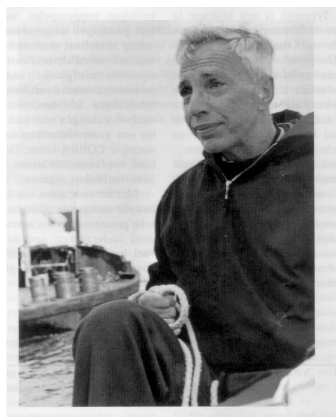
The *star product* is the Weyl correspondent of the Hilbert space operator product.
H Weyl (1927), J von Neumann (1931), E Wigner (1932), H Groenewold (1946), J Moyal (1949)

$$f\star g=\int\frac{dx_1dp_1}{2\pi\left(\hbar/2\right)}\int\frac{dx_2dp_2}{2\pi\left(\hbar/2\right)}\ f\left(x+x_1,p+p_1\right)\ g\left(x+x_2,p+p_2\right)\ \exp\left(\frac{i}{\hbar/2}\left(x_1p_2-x_2p_1\right)\right)\qquad\text{JvN/HW/EW/HG}$$

$$\begin{aligned}x_1p_2-x_2p_1&=\text{Area}\left(\text{1,2 phase-space parallelogram}\right)\\ \hbar/2&=\text{Planck Area}=\min\left(\Delta x\Delta p\right)\end{aligned}$$

$$f \star g = f\,(x,p)\ \exp\left(\overleftarrow{\partial_x}\,\frac{i\hbar}{2}\,\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\,\frac{i\hbar}{2}\,\overrightarrow{\partial_x}\right)\ g\,(x,p) \qquad \text{EW/HG/JM}$$

$$\begin{aligned} f \star g &= f\left(x + \frac{1}{2} i\hbar\,\overrightarrow{\partial_p}\ ,\ p - \frac{1}{2} i\hbar\,\overrightarrow{\partial_x}\right)\ g\,(x,p) \\ &= f\,(x,p)\ g\left(x - \frac{1}{2} i\hbar\overleftarrow{\partial_p}\ ,\ p + \frac{1}{2} i\hbar\,\overleftarrow{\partial_x}\right) \\ &= f\left(x + \frac{1}{2} i\hbar\,\overrightarrow{\partial_p}\ ,\ p\right)\ g\left(x - \frac{1}{2} i\hbar\overleftarrow{\partial_p}\ ,\ p\right) \\ &= f\left(x, p - \frac{1}{2} i\hbar\,\overrightarrow{\partial_x}\right)\ g\left(x, p + \frac{1}{2} i\hbar\,\overleftarrow{\partial_x}\right) \end{aligned}$$

Exercise 1 *WFs as star products (Braunss)*

$$f_{12}(x,p) = \psi_1(x) \star \delta(p) \star \psi_2^*(x)$$

Exercise 2 *Non-commutativity*

$$\begin{gathered} e^{ax+bp} \star e^{Ax+Bp} = e^{(a+A)x+(b+B)p}\ e^{(aB-bA)i\hbar/2} \\ \neq \\ e^{Ax+Bp} \star e^{ax+bp} = e^{(a+A)x+(b+B)p}\ e^{(Ab-Ba)i\hbar/2} \end{gathered}$$

Exercise 3 *Associativity*

$$\begin{aligned} &\left(e^{ax+bp} \star e^{Ax+Bp}\right) \star e^{\alpha x+\beta p} \\ &= e^{(a+A+\alpha)x+(b+B+\beta)p}\ e^{(aB-bA+a\beta-b\alpha+A\beta-B\alpha)i\hbar/2} \\ &= e^{ax+bp} \star \left(e^{Ax+Bp} \star e^{\alpha x+\beta p}\right) \end{aligned}$$

Exercise 4 *Trace properties (a.k.a. Lone Star Lemma)*

$$\int dxdp\, f \star g = \int dxdp\, f\, g = \int dxdp\, g\, f = \int dxdp\, g \star f$$

Establish this by using the RHS in the first exercise for purely imaginary a, b, A, and B to obtain Dirac deltas which eliminate the non-commutative phase. Or equivalently, just use the integral form of $f \star g$.

Exercise 5 *Gaussians. The integral form of $\star$ is particularly useful to show, for $a, b \geq 0$,*

$$\exp\left(-\frac{a}{\hbar}\left(x^2+p^2\right)\right) \star \exp\left(-\frac{b}{\hbar}\left(x^2+p^2\right)\right) = \frac{1}{1+ab}\exp\left(-\frac{a+b}{(1+ab)\,\hbar}\left(x^2+p^2\right)\right)\ .$$

Hence such Gaussians are exceptional and $\star$ commute with one another.

Fundamental pure-state conditions

Pure-state Wigner functions must obey a *projection* condition. If the normalization is set to the standard value

$$\iint_{-\infty}^{+\infty} dxdp\, f(x,p) = 1$$

then the function corresponds to a pure state if and only if

$$f = (2\pi\hbar)\ f \star f$$

These statements correspond to the pure-state density operator conditions: $Tr(\widehat{\boldsymbol{\rho}}) = 1$ and $\widehat{\boldsymbol{\rho}} = \widehat{\boldsymbol{\rho}}\,\widehat{\boldsymbol{\rho}}$, respectively.

If both of the above are true, then f describes an allowable pure state for a quantized system. Otherwise not.[4] You can easily satisfy only one out of these two conditions, but not the other, with f not a pure state.

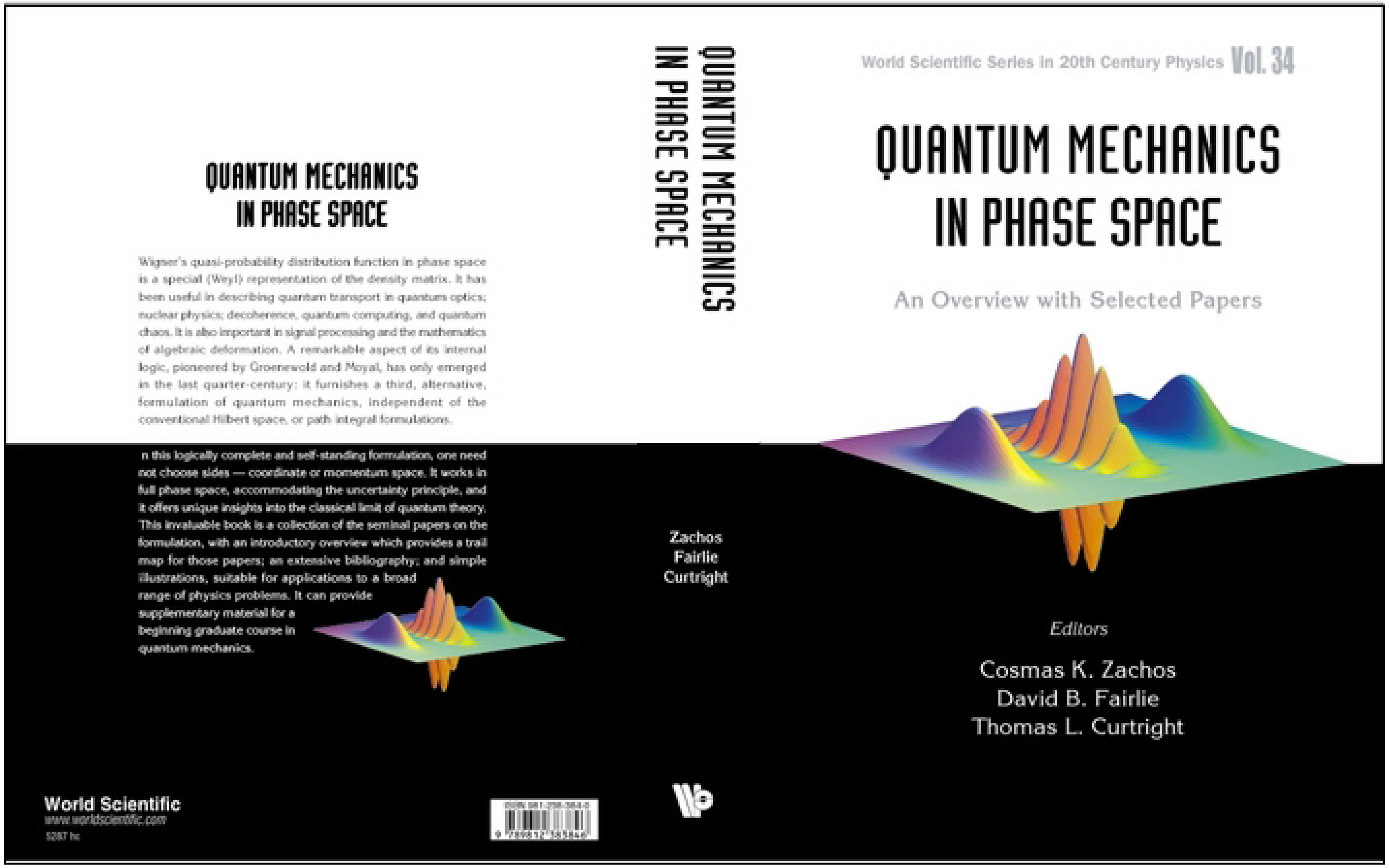
Without drawing on the Hilbert space formulation, it may at first seem to be rather remarkable that explicit WFs actually satisfy the projection condition (cf. the above Gaussian example, for the only situation where it works, $a^2 = b^2 = 1$, i.e. $\exp(-(x^2+p^2)/\hbar))$. However, if the WFs are known to be $\star$ eigenfunctions with non-vanishing eigenvalue of some phase-space function with a non-degenerate spectrum of eigenvalues, they have no choice but to obey $f \star f \propto f$ as a consequence of associativity.

[4] You may of course select a different standard norm for pure states, say $\iint_{-\infty}^{+\infty} dxdp\, f(x,p) = N$, and if you do, the projection condition is likewise changed to $Nf = (2\pi\hbar)\ f \star f$. (Actually, this seemingly trivial change is important for systems with a continuum of energies.) In any case, $f \star f \propto f$ is necessary for a pure state.

We emphasize that there is no need to deal with wave functions or Hilbert space states. The WFs may be constructed directly on the phase-space. For *real* Hamiltonians, energy eigenstates are obtained as (real) solutions of the $\star$-*genvalue* equations (Fairlie 1964, Bayen et al. 1978, or more recently *hep-th/9711183*):

$$H \star f = E\, f = f \star H$$

Also see David's recent book on QM in Phase Space ... ahem!


QUANTUM MECHANICS
IN PHASE SPACE
Wigner's quasi-probability distribution function in phase space is a special (Weyl) representation of the density matrix. It has been useful in describing quantum transport in quantum optics; nuclear physics; decoherence, quantum computing, and quantum chaos. It is also important in signal processing and the mathematics of algebraic deformation. A remarkable aspect of its internal logic, pioneered by Groenewold and Moyal, has only emerged in the last quarter-century: it furnishes a third, alternative, formulation of quantum mechanics, independent of the conventional Hilbert space, or path integral formulations.
n this logically complete and self-standing formulation, one need not choose sides — coordinate or momentum space. It works in full phase space, accommodating the uncertainty principle, and it offers unique insights into the classical limit of quantum theory. This invaluable book is a collection of the seminal papers on the formulation, with an introductory overview which provides a trail map for those papers; an extensive bibliography; and simple illustrations, suitable for applications to a broad range of physics problems. It can provide supplementary material for a beginning graduate course in quantum mechanics.
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An Overview with Selected Papers
Editors
Cosmas K. Zachos
David B. Fairlie
Thomas L. Curtright

The simple harmonic oscillator (SHO) To illustrate all this, consider the SHO ($m = 1, \omega = 1$) with

$$H = \frac{1}{2}\left(p^2 + x^2\right)$$

The above equations are *partial* differential equations

$$H \star f = \frac{1}{2}\left(\left(p - \frac{1}{2} i\hbar\, \partial_x\right)^2 + \left(x + \frac{1}{2} i\hbar\, \partial_p\right)^2\right) f = Ef$$
$$f \star H = \frac{1}{2}\left(\left(p + \frac{1}{2} i\hbar\, \partial_x\right)^2 + \left(x - \frac{1}{2} i\hbar\, \partial_p\right)^2\right) f = Ef$$

But if we subtract (or take the imaginary part)

$$\left(p\partial_x - x\partial_p\right) f = 0 \qquad \Rightarrow \qquad f\left(x, p\right) = f\left(x^2 + p^2\right)$$

So $H \star f = Ef = f \star H$ becomes a single *ordinary* differential equation.

That's Laguerre  not Hermite! 

There are integrable solutions if and only if $E = (n + 1/2)\,\hbar$, $n = 0, 1, \cdots$ for which

$$f_n(x,p) = \frac{(-1)^n}{\pi\hbar} L_n\left(\frac{x^2+p^2}{\hbar/2}\right) e^{-\left(x^2+p^2\right)/\hbar}$$
$$L_n(z) = \frac{1}{n!} e^z \frac{d^n}{dz^n}\left(z^n e^{-z}\right)$$

The normalization is chosen to be the standard one $\iint_{-\infty}^{+\infty} dxdp\, f_n(x,p) = 1$. Except for the $n = 0$ ground state Wiggie (Gaussian) these f's change sign on the xp-plane. For example:

$$L_0(z) = 1\ , \qquad L_1(z) = 1 - z\ , \qquad L_2(z) = 1 - 2z + \frac{1}{2}z^2\ , \qquad \text{etc.}$$

Exercise 6 *Using the integral form of the $\star$ product, it is now easy to check these pure states are $\star$ orthogonal*

$$(2\pi\hbar)\ f_n \star f_k = \delta_{nk}\, f_n$$

Exercise 7 *This is even more transparent using $\star$ raising/lowering operations to write*[5]

$$\begin{aligned} f_n &= \frac{1}{n!}\,(a^*\star)^n\ f_0\ (\star a)^n \\ &= \frac{1}{\pi\hbar n!}\,(a^*\star)^n\ e^{-\left(x^2+p^2\right)/\hbar}\ (\star a)^n \end{aligned}$$

where a is the usual linear combination $a \equiv \frac{1}{\sqrt{2\hbar}}(x+ip)$, and a^* is just its ordinary complex conjugate $a^* \equiv \frac{1}{\sqrt{2\hbar}}(x-ip)$, with $a\star a^* - a^*\star a = 1$, and $a\star f_0 = 0 = f_0\star a^*$ (cf. coherent states).

Exercise 8 *Non-diagonal WFs are equally easy to construct in terms of associated Laguerre polynomials*

$$f_{nk} = e^{i(n-k)\ \arctan(p/x)}\,\frac{(-1)^k}{\pi\hbar}\,(\frac{x^2+p^2}{\hbar/2})^{(n-k)/2}\,L_k^{n-k}\left(\frac{x^2+p^2}{\hbar/2}\right)\ e^{-\left(x^2+p^2\right)/\hbar}$$

either as direct solutions to

$$\begin{aligned} H\star f_{nk} &= E_n\, f_{nk} \\ f_{nk}\star H &= E_k\, f_{nk} \end{aligned}$$

or in terms of raising/lowering $\star$ operations

$$f_{nk} = \frac{1}{\sqrt{n!k!}}\,(a^*\star)^n\ f_0\ (\star a)^k$$

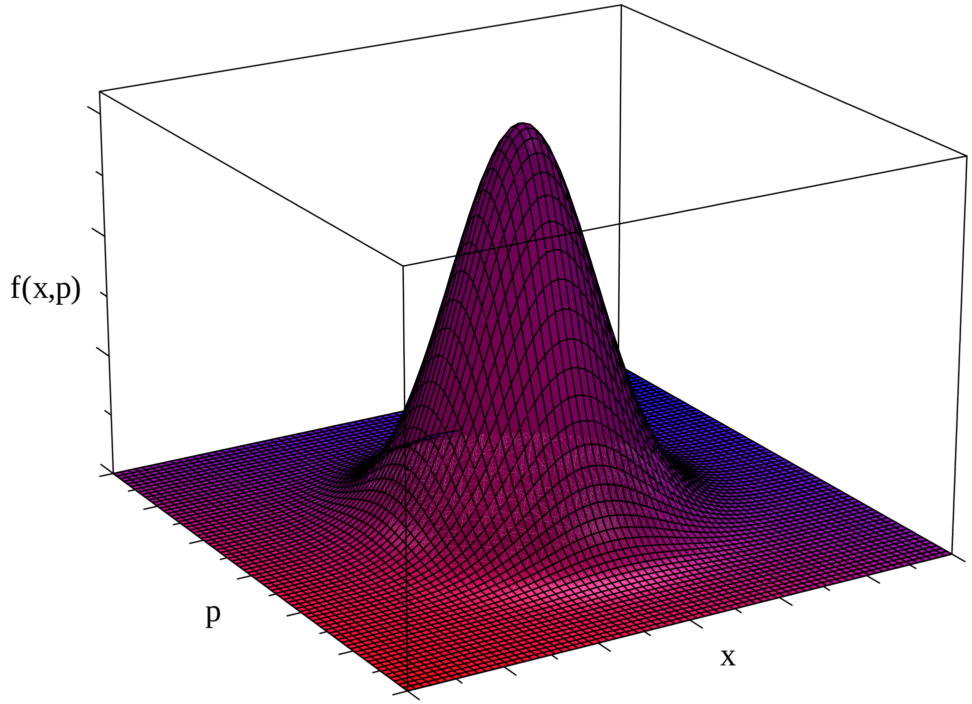
Exercise 9 *These are also $\star$ orthogonal*

$$\delta_{lm} f_{nk} = (2\pi\hbar)\ f_{nm}\star f_{lk}$$

as well as complete ($f_{nm} = f^*_{mn}$)

$$(2\pi\hbar)\sum_{m,n} f_{mn}\left(x_1,p_1\right)\ f_{nm}\left(x_2,p_2\right) = \delta\left(x_1-x_2\right)\ \delta\left(p_1-p_2\right)$$

[5] Note that the earlier exercise giving the star composition law of Gaussians immediately yields the projection property of the SHO ground state Wiggie $f_0 = (2\pi\hbar)\, f_0\star f_0$.



SHO $n = 0$ Wigner function

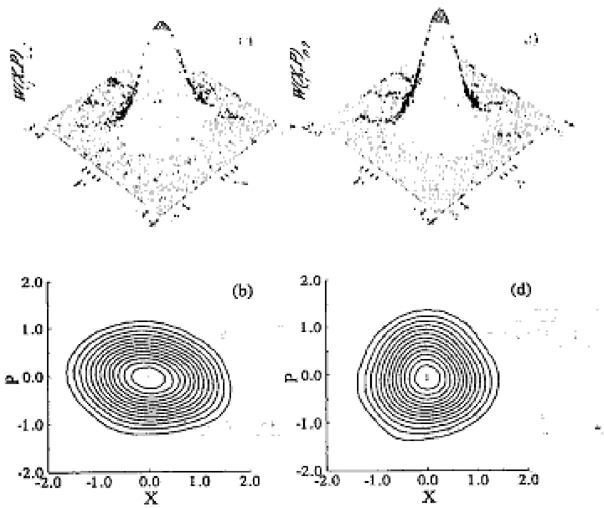


FIG. 1. Measured Wigner distributions for (a),(b) a squeezed state and (c),(d) a vacuum state, viewed in 3D and as contour plots, with equal numbers of constant-height contours. Squeezing of the noise distribution is clearly seen in (b).

Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum

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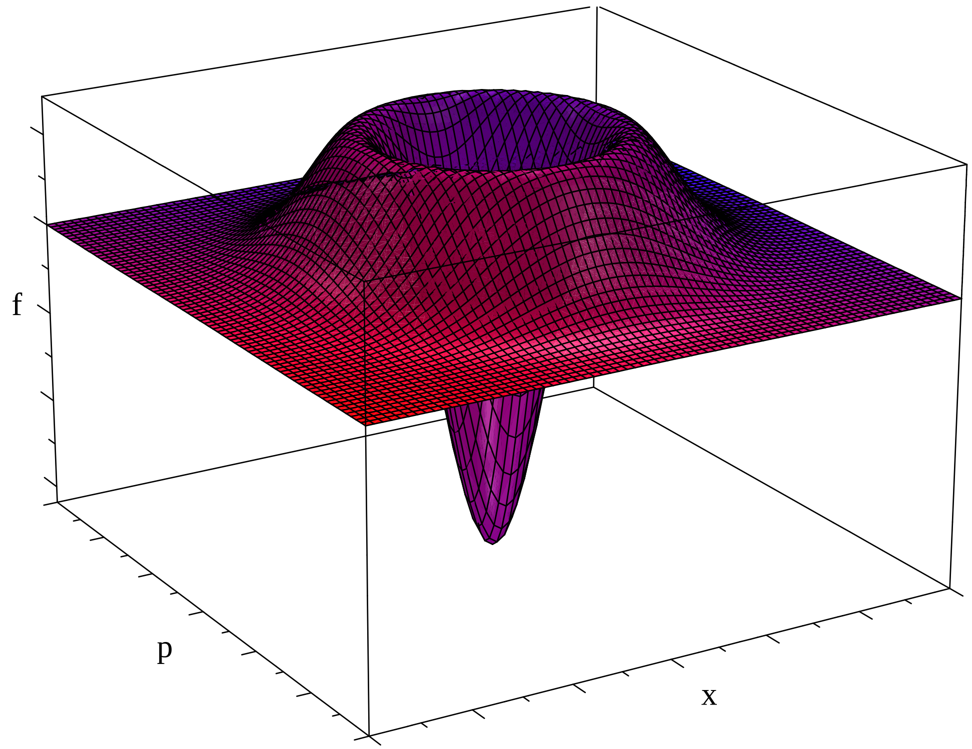
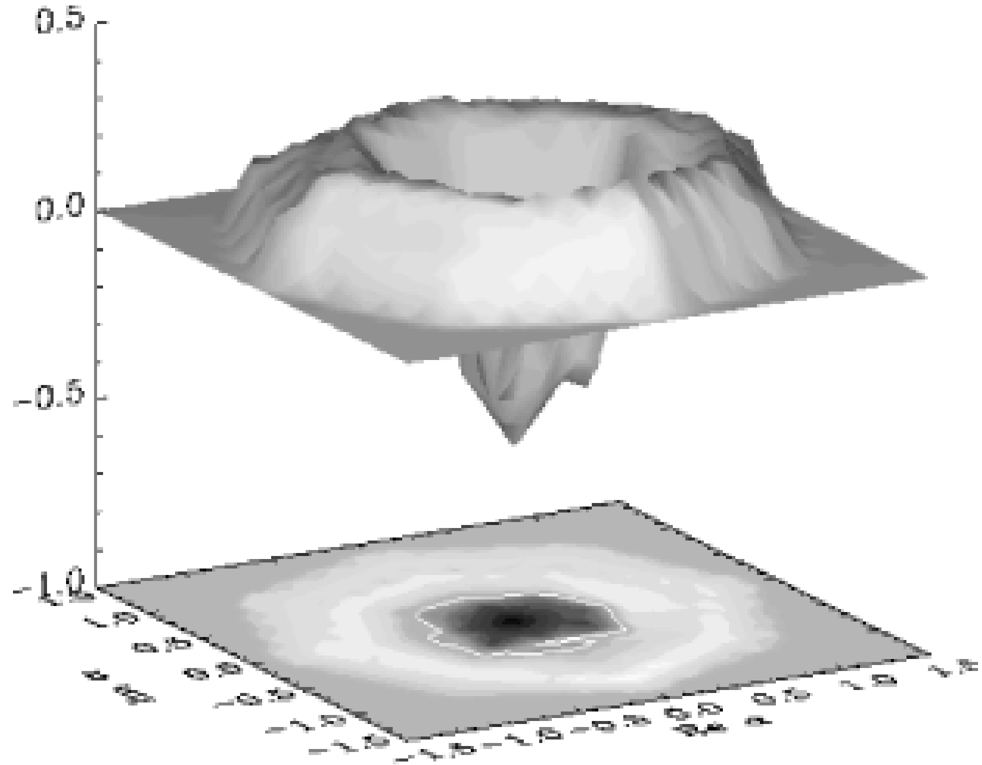
Phys. Rev. Lett. 70, 1244–1247 (1993)
(Received 16 November 1992)

We have measured probability distributions of quadrature-field amplitude for both vacuum and quadrature-squeezed states of a mode of the electromagnetic field. From these measurements we demonstrate the technique of optical homodyne tomography to determine the Wigner distribution and the density matrix of the mode. This provides a complete quantum mechanical characterization of the measured mode.



SHO $n = 1$ Wigner function


0.5
0.0
−0.5
−1.0

Experimental Determination of the Motional Quantum State of a Trapped Atom

D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland
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Phys. Rev. Lett. **77**, 4281–4285 (1996)
(Received 11 July 1996)

We reconstruct the density matrices and Wigner functions for various quantum states of motion of a harmonically bound $^9Be^+$ ion. We apply coherent displacements of different amplitudes and phases to the input state and measure the number state populations. Using novel reconstruction schemes we independently determine both the density matrix in the number state basis and the Wigner function. These reconstructions are sensitive indicators of decoherence in the system.

End of tutorial ... back to the matters at hand.