TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

A non-Hermitian two-mode Bose-Hubbard system

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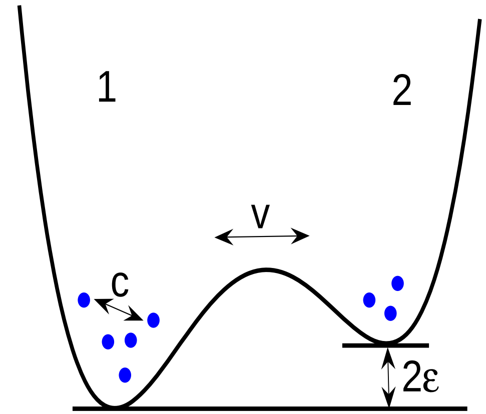
DFG Graduierten-kolleg 792

6th International Workshop on Pseudo Hermitian Hamiltonians In Quantum Physics (16th-18th of July 2007, City University London)

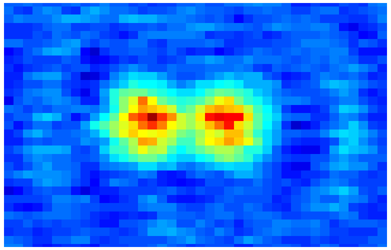
A two-mode Bose-Hubbard system

$$\hat{H} = \varepsilon(\hat{n}_1 - \hat{n}_2) + v(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + c\left(\hat{n}_1^2 + \hat{n}_2^2\right)$$

on-site energy — tunneling — on-site interaction



Bose-Einstein Condensate in a double well trap



experiments (M. Oberthaler et al.):

- Josephson-oscillations
- Self trapping

A two-mode Bose-Hubbard system

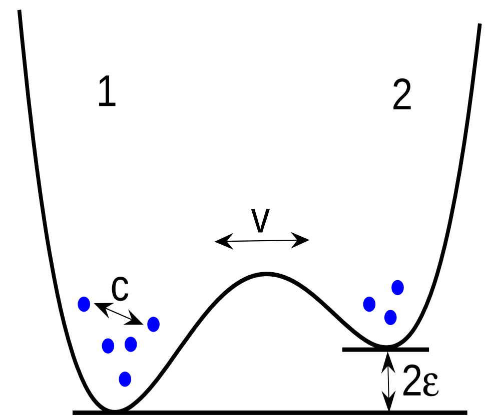
$$\hat{H} = (-i\gamma + \varepsilon)(\hat{n}_1 - \hat{n}_2) + v(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + c\left(\hat{n}_1^2 + \hat{n}_2^2\right)$$

on-site energy

non-Hermiticity

tunneling

on-site interaction

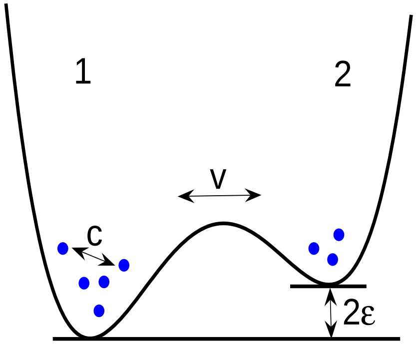


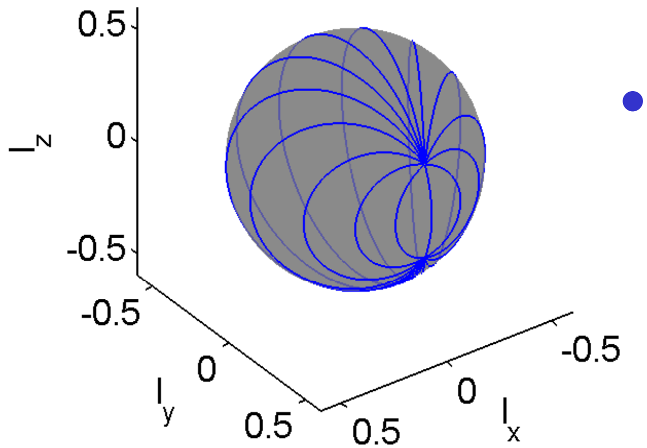
Modelling a sink in one of the wells, a source in the other

M. Hiller, T. Kottos, A. Ossipov, Phys. Rev. A 73, 063625 (2006)

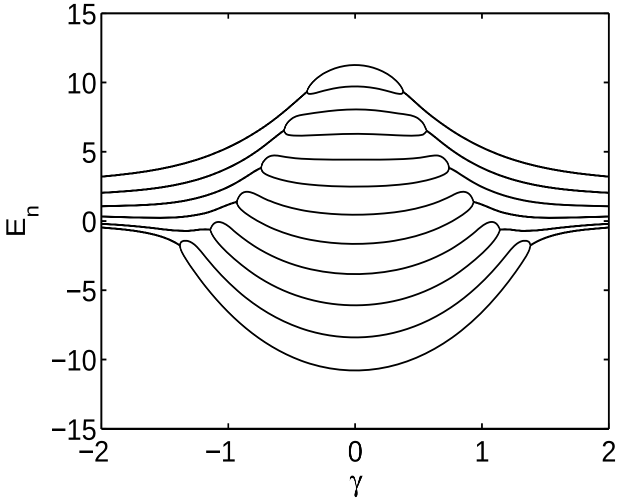
Outline

- The hermitian Bose-Hubbard model:
  many-particle and mean-field
  Self trapping transition





- The non-Hermitian case:
  The limit of vanishing interaction



- Interaction and non-Hermiticity:
  bifurcations of resonances

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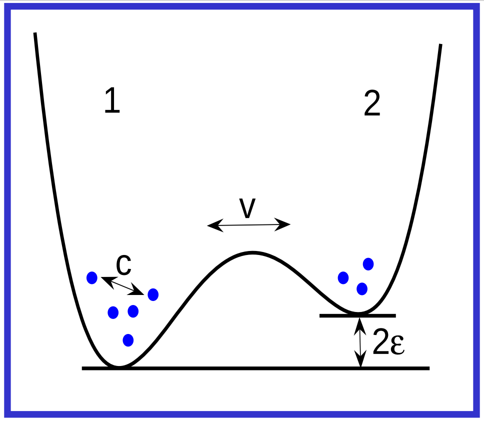

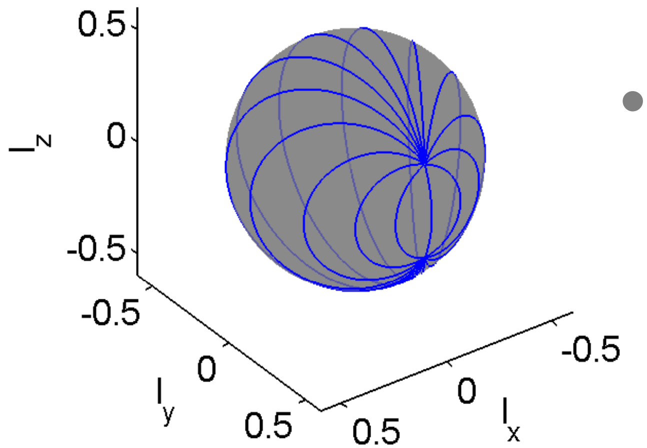

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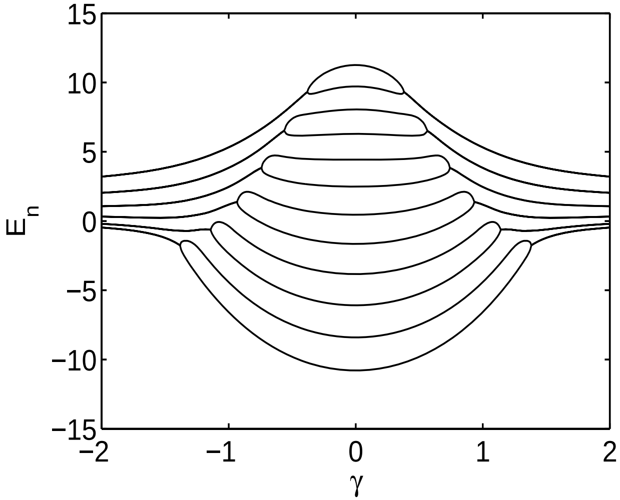


- Interaction and non-Hermiticity:
  bifurcations of resonances

Many-particle Hamiltonian

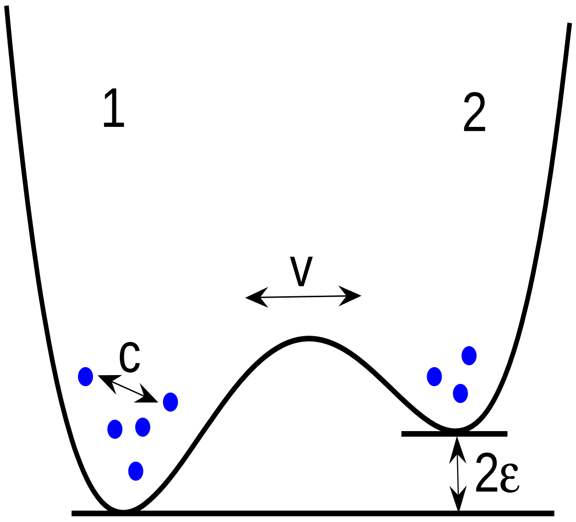
$$\hat{H} = \varepsilon(\hat{n}_1 - \hat{n}_2) + v(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) - \frac{c}{2}(\hat{n}_1 - \hat{n}_2)^2$$

Jordan-Schwinger transformation:

$$\hat{L}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)$$

$$\hat{L}_y = \frac{1}{2\mathrm{i}}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$$

$$\hat{L}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)$$



$$\hat{H} = 2\varepsilon \hat{L}_z + 2v\hat{L}_x - 2c\hat{L}_z^2$$

Mean-field / classical approximation

Replace operators by c-numbers:

$$\hat{a}_j^\dagger \to \sqrt{N_s}\psi_j^* \quad \hat{a}_j \to \sqrt{N_s}\psi_j \qquad N_s = N + 1$$

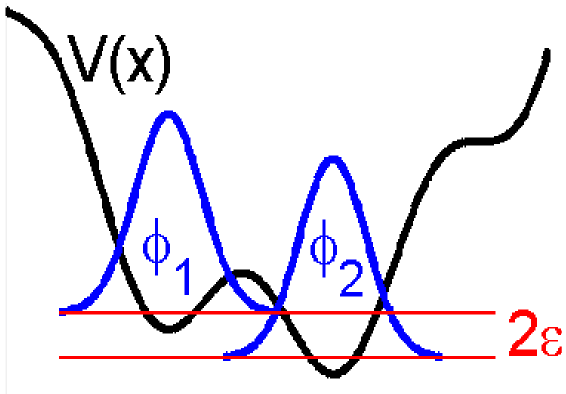
Attention: Start from a symmetric representation

$$\frac{1}{2}(\hat{a}_j^\dagger \hat{a}_j + \hat{a}_j \hat{a}_j^\dagger) \to N_s |\psi_j|^2$$

single particle wave function

$$\psi(x,t) = \psi_1(t)\phi_1(x) + \psi_2(t)\phi_2(x)$$

with $|\psi_1(t)|^2 + |\psi_2(t)|^2 = 1$



Meanfield dynamics

Hamiltonian $\longrightarrow$ Classical Hamiltonian function

$$\mathcal{H} = \varepsilon(|\psi_1|^2 - |\psi_2|^2) + v(\psi_1^*\psi_2 + \psi_2^*\psi_1) - \frac{c}{2}\underbrace{N_s\left(|\psi_1|^2 - |\psi_2|^2\right)^2}_{\kappa}$$

Canonical equations of motion:

$$\mathrm{i}\hbar\dot{\psi}_j = \frac{\partial\mathcal{H}}{\partial\psi_j^*} \qquad \mathrm{i}\hbar\dot{\psi}_j^* = -\frac{\partial\mathcal{H}}{\partial\psi_j}$$

Nonlinear Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\varepsilon - c\kappa & v\\ v & -\varepsilon + c\kappa\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Mean-field Bloch representation

$$\langle \hat{L}_j \rangle \to N_s l_j, \quad \langle \hat{L}_j \hat{L}_k + \hat{L}_k \hat{L}_j \rangle \approx 2 \langle \hat{L}_j \rangle \langle \hat{L}_k \rangle$$

$$\hat{L}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \qquad l_x = \frac{1}{2}(\psi_1^* \psi_2 + \psi_2^* \psi_1)$$

$$\hat{L}_y = \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \qquad l_y = \frac{1}{2i}(\psi_1^* \psi_2 - \psi_2^* \psi_1)$$

$$\hat{L}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \qquad l_z = \frac{1}{2}(\psi_1^* \psi_1 - \psi_2^* \psi_2)$$

Equations of motion:

$$\frac{d}{dt}\hat{L}_j = i[\hat{H}, \hat{L}_j] \quad \to \quad \frac{d}{dt} l_j = i\{\mathcal{H}, l_j\}_{\psi_j, \psi_j^*}$$

Nonlinear Bloch equations

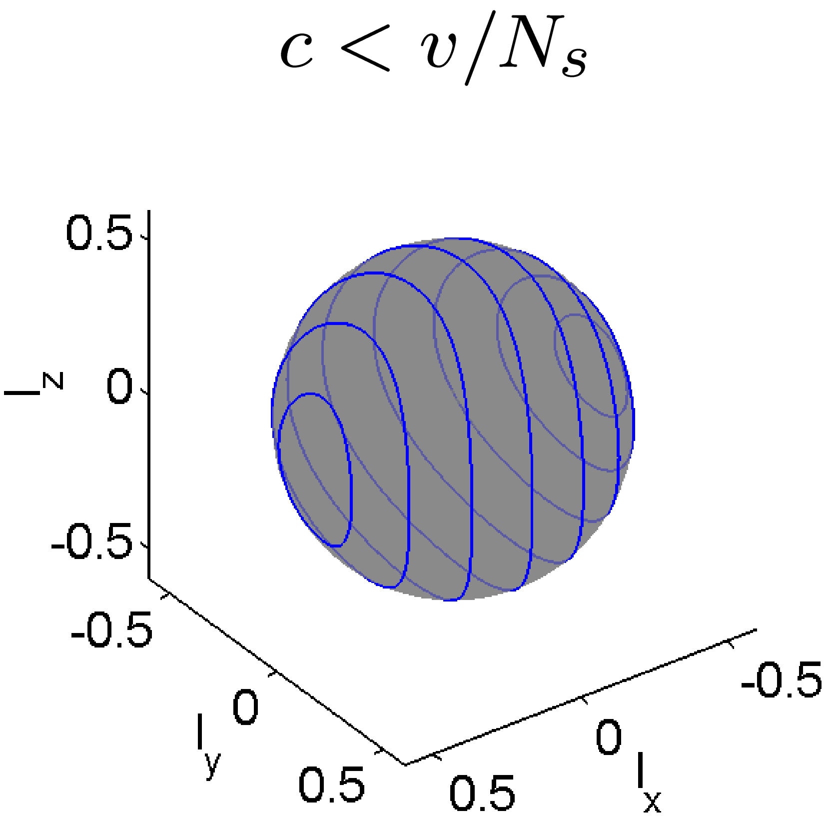
$$\frac{d}{dt}l_x = -2\varepsilon l_y + 2cN_s l_y l_z$$
$$\frac{d}{dt}l_y = 2\varepsilon l_x - 2v l_z - 2cN_s l_x l_z$$
$$\frac{d}{dt}l_z = 2v l_y$$

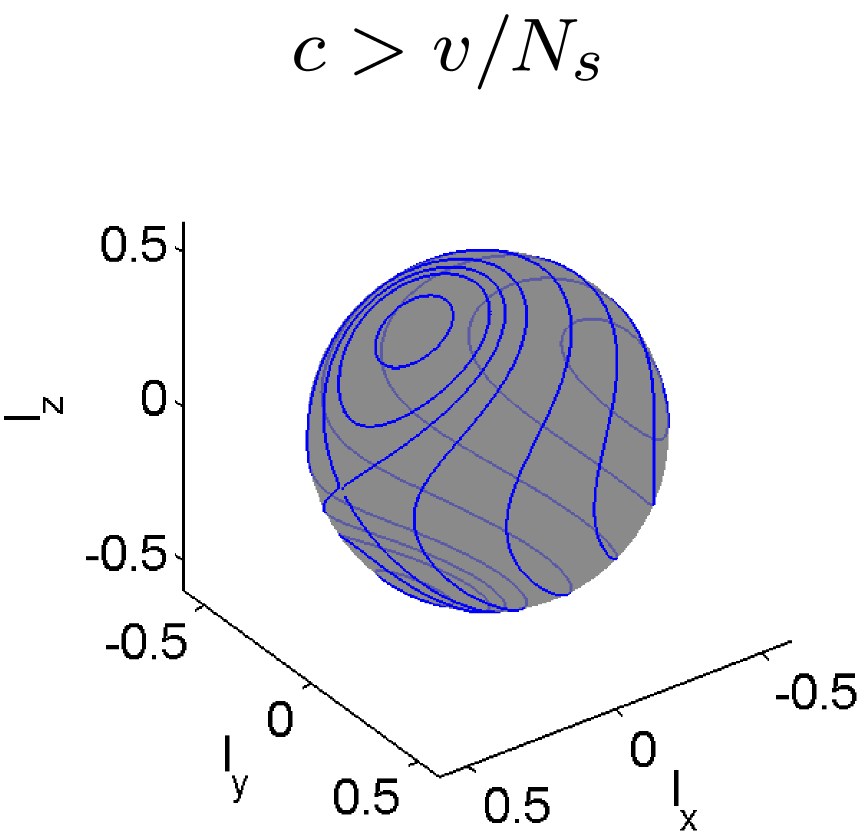
Constant of motion: normalization

$$\sum_k l_k^2 = \left(\frac{1}{2}\right)^2$$

$\Rightarrow$ motion restricted to a sphere

Classical phase space





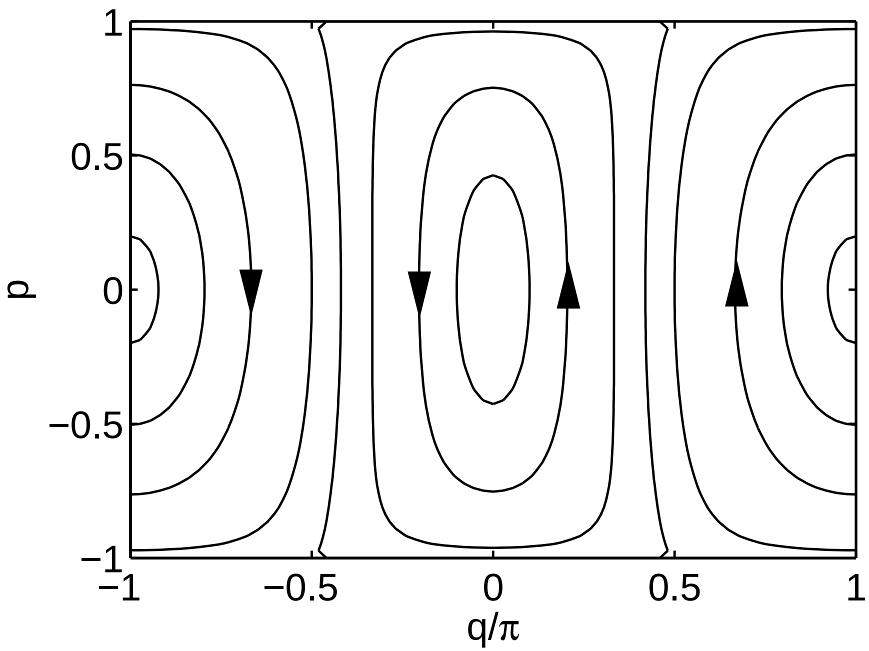
Above a critical interaction strength: Selftrapping

Classical phase space

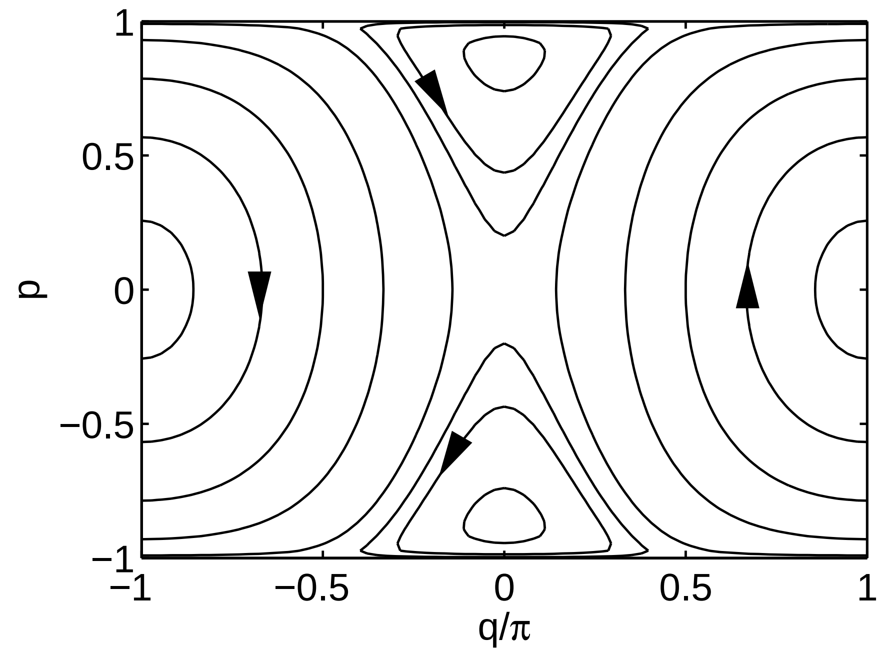
New representation:

$$\vec{l} = \begin{pmatrix} \sqrt{1-p^2}\cos(q) \\ \sqrt{1-p^2}\sin(q) \\ p/2 \end{pmatrix}$$

$c < v/N_s$

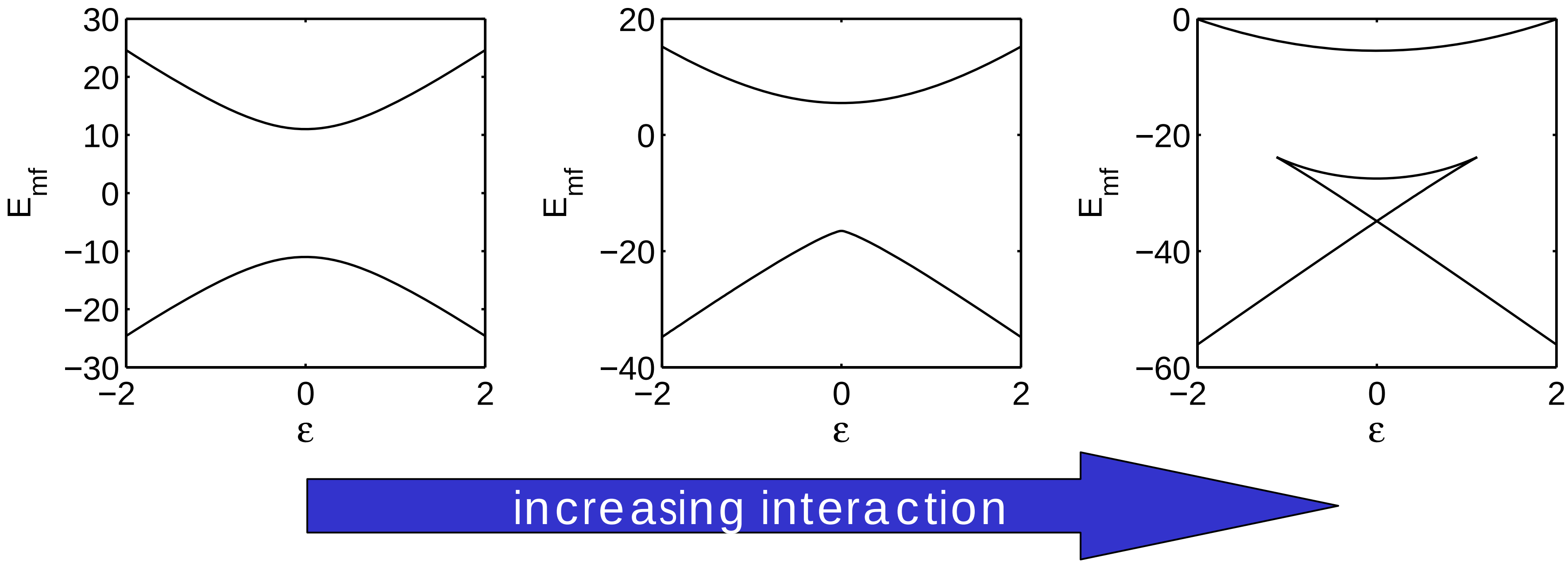


$c > v/N_s$



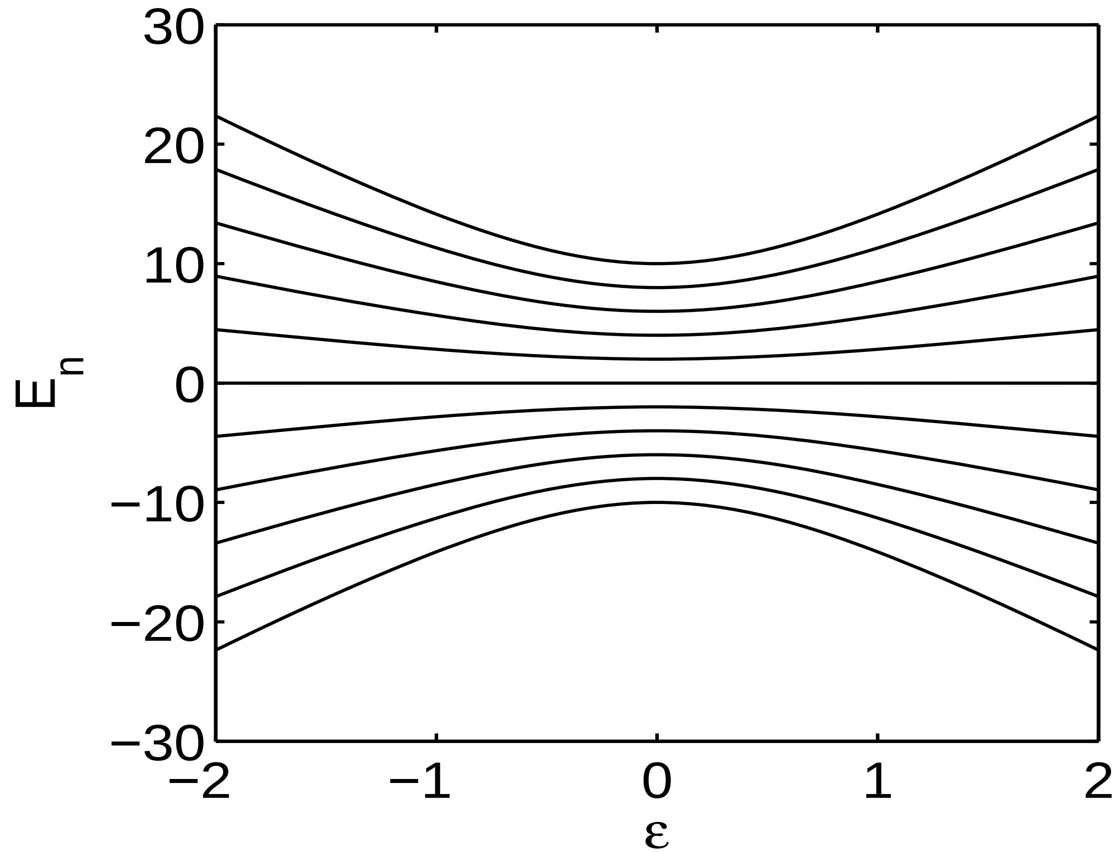
Mean-field energies

- Mean-field energies: stationary values of the Hamiltonian function
- Depending on the interaction strength: 2 up to 4 stationary points for each parameter set



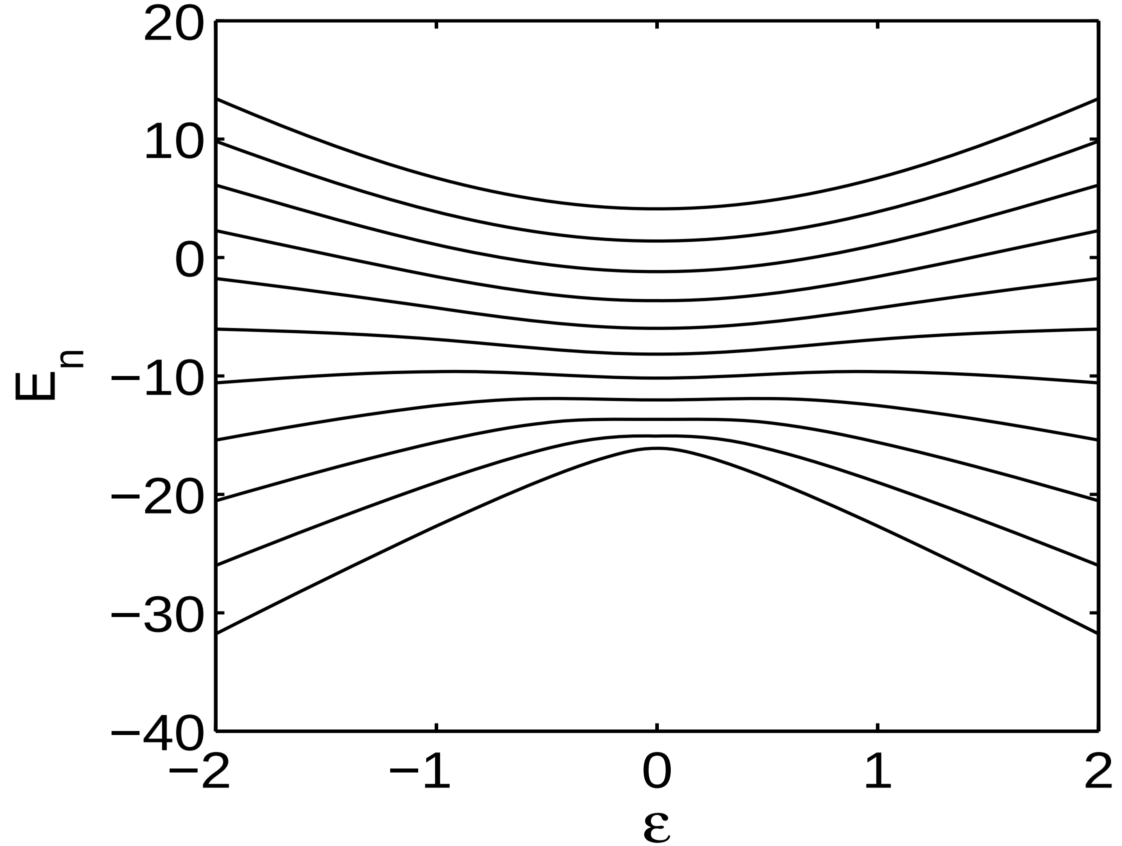
Eigenenergies for N=10 particles

$c = 0$



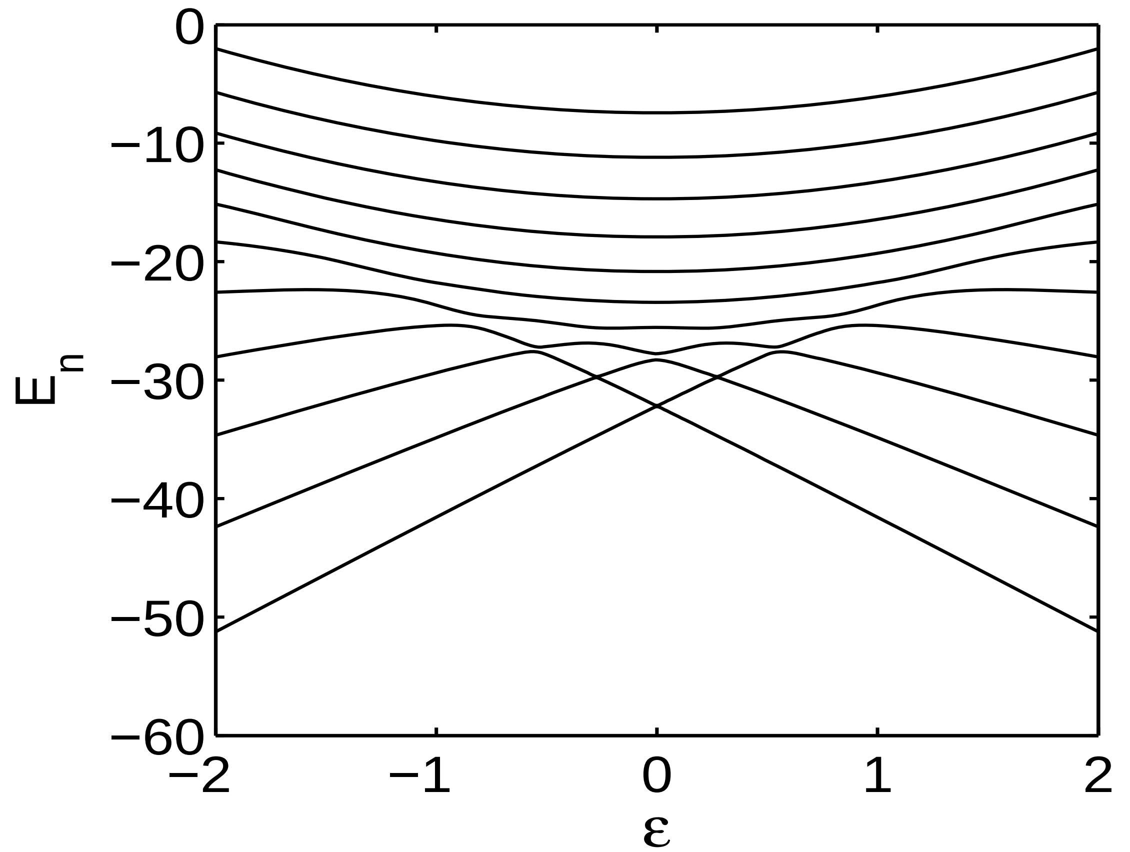
Eigenenenergies for N=10 particles

$$c = v/N_s$$

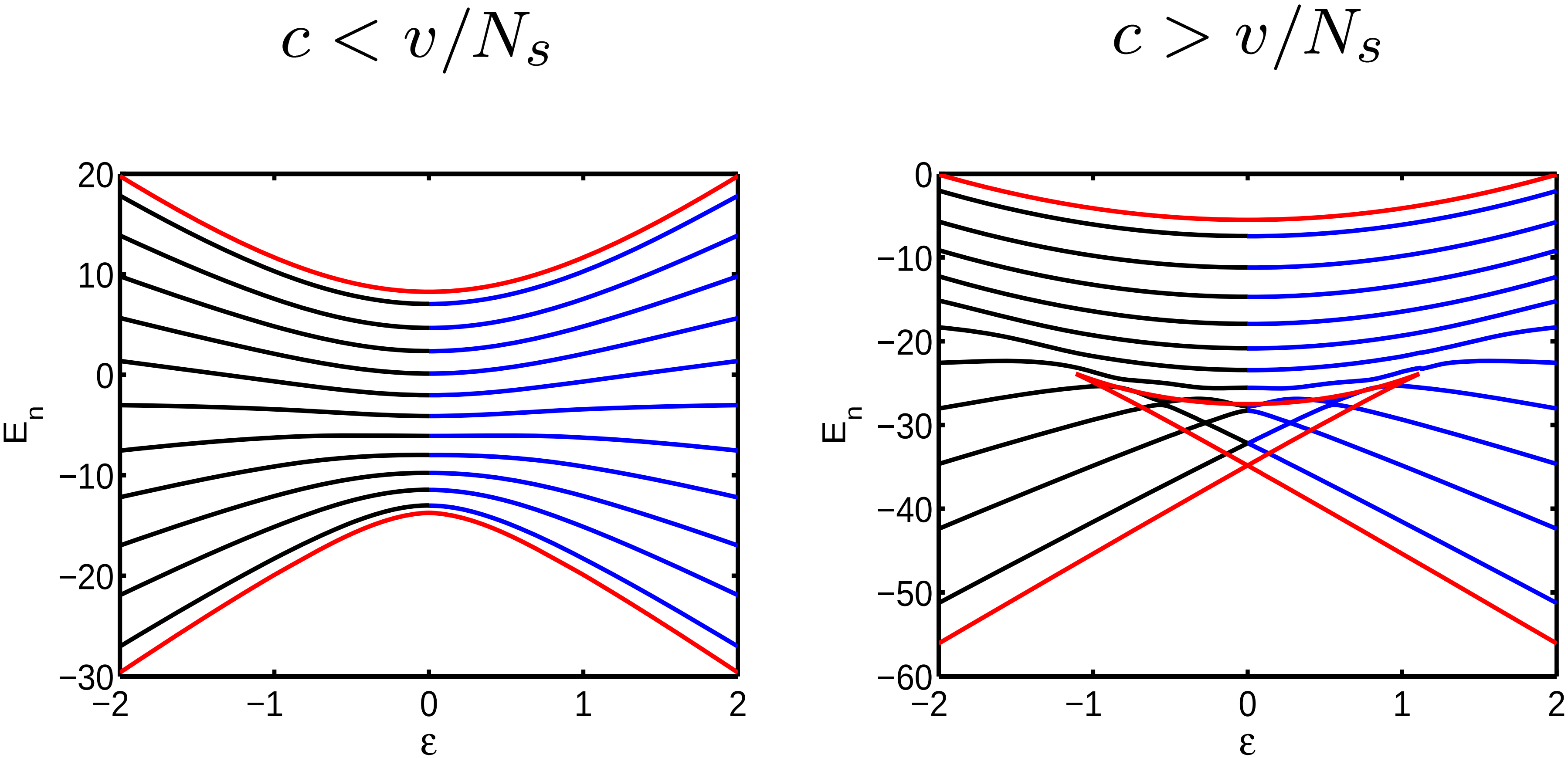


Eigenenergies for N=10 particles

$$c > v/N_s$$



Exact, classical and semiclassical eigenvalues for N=10 particles



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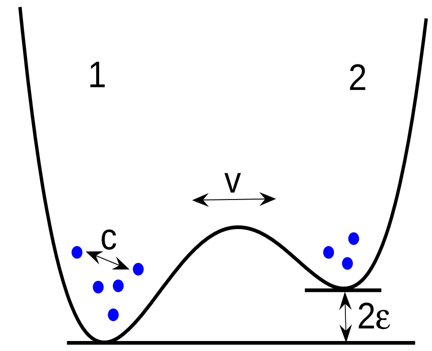

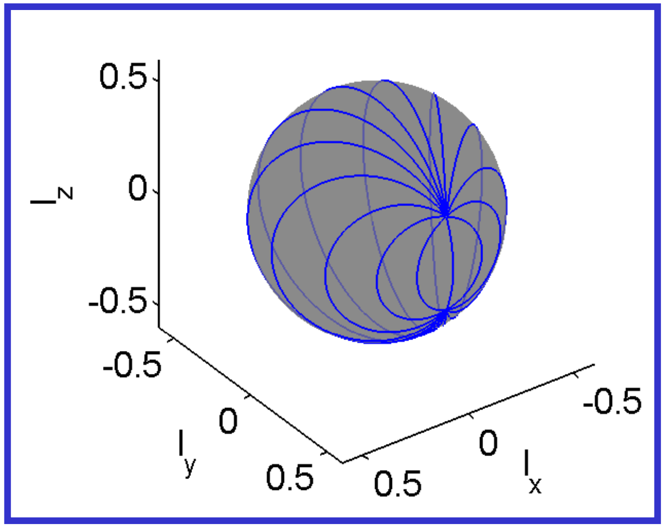

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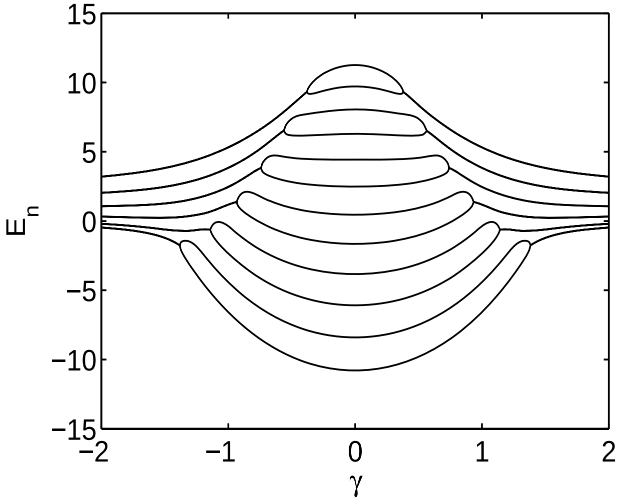


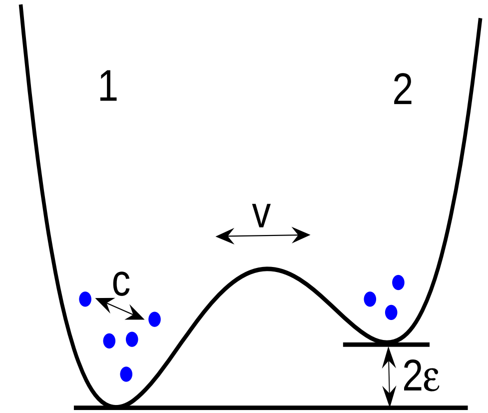
- Interaction and non-Hermiticity:
  bifurcations of resonances

A non-Hermitian Bose-Hubbard system

$$\hat{H} = (\varepsilon - i\gamma)(\hat{n}_1 - \hat{n}_2) + v(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) - \frac{c}{2}(\hat{n}_1 - \hat{n}_2)^2$$

$\mathcal{T}$: Complex conjugation

$\mathcal{P}$: Interchange modes 1 and 2

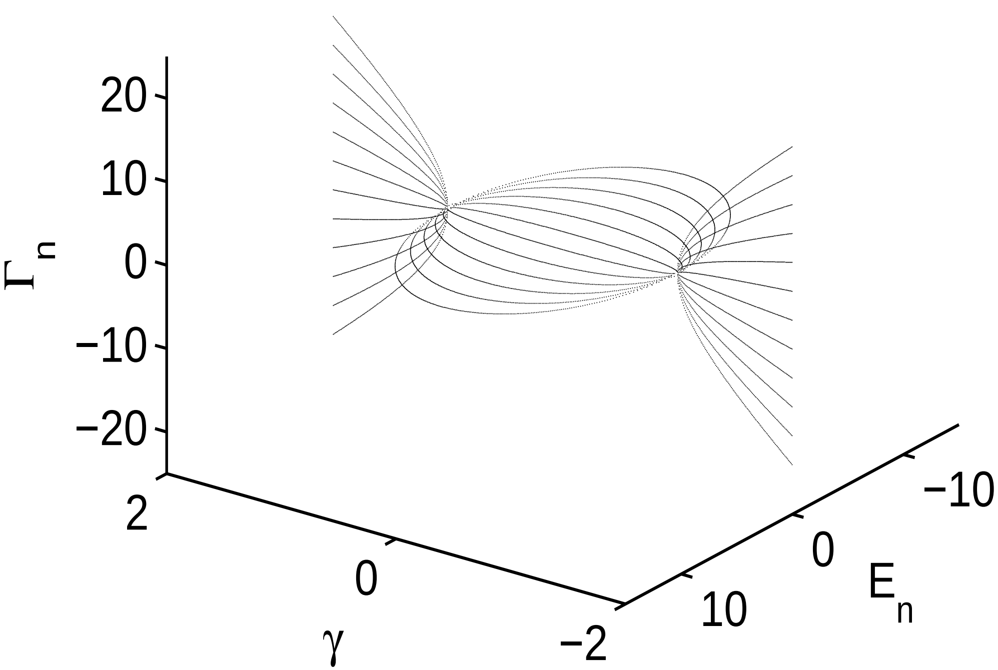


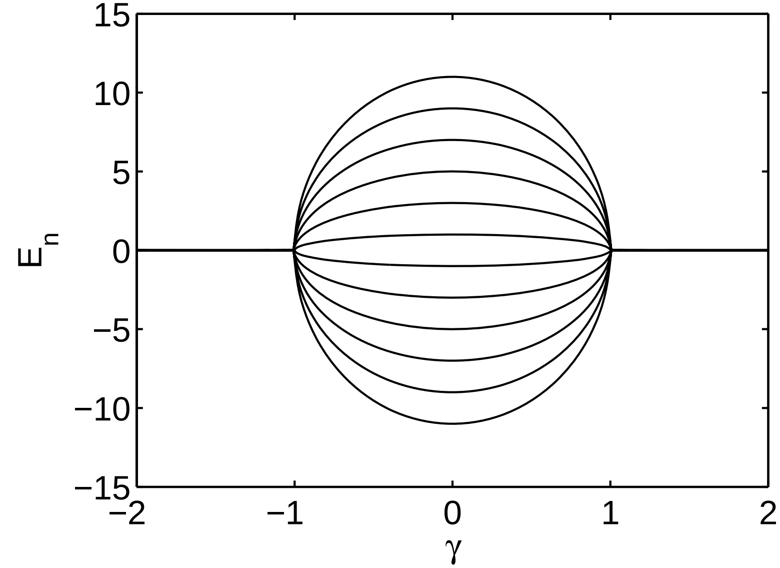
- $\varepsilon = 0$: $\hat{H}$ is $\mathcal{PT}$ symmetric
- Eigenvalues for vanishing interaction: $c = 0$

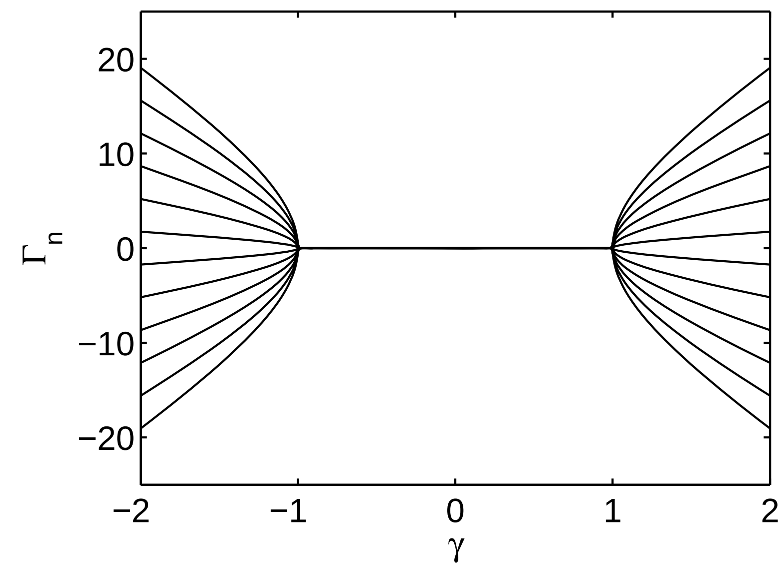
$$\mathcal{E}_n = E_n - \mathrm{i}\Gamma_n = (2n - N)\sqrt{v^2 - \gamma^2}, \quad n = 0 \ldots N$$

Eigenvalues for N=11 particles

$$c = 0$$







The non-Hermitian mean-field system

Replacing ladder operators by c-numbers

$$\hat{a}_j^\dagger \to \sqrt{N_s}\psi_j^* \quad \hat{a}_j \to \sqrt{N_s}\psi_j \quad \frac{1}{2}(\hat{a}_j^\dagger \hat{a}_j + \hat{a}_j \hat{a}_j^\dagger) \to N_s|\psi_j|^2$$

yields a Hamiltonian function

$$\mathcal{H} = (\varepsilon - \mathrm{i}\gamma)(\psi_1^*\psi_1 - \psi_2^*\psi_2) + v(\psi_1^*\psi_2 + \psi_2^*\psi_1)$$

Bloch representation:

$$\langle \hat{L}_j \rangle = \frac{\langle \Psi | \hat{L}_j | \Psi \rangle}{\langle \Psi | \Psi \rangle} \to N_s l_j$$

Mean-field Bloch representation

$$\langle \hat{L}_j \rangle = \frac{\langle \Psi | \hat{L}_j | \Psi \rangle}{\langle \Psi | \Psi \rangle} \rightarrow N_s l_j$$

$$\hat{L}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \qquad l_x = \frac{1}{2}\frac{(\psi_1^* \psi_2 + \psi_2^* \psi_1)}{(\psi_1^* \psi_1 + \psi_2^* \psi_2)}$$

$$\hat{L}_y = \frac{1}{2\mathrm{i}}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \qquad l_y = \frac{1}{2\mathrm{i}}\frac{(\psi_1^* \psi_2 - \psi_2^* \psi_1)}{(\psi_1^* \psi_1 + \psi_2^* \psi_2)}$$

$$\hat{L}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \qquad l_z = \frac{1}{2}\frac{(\psi_1^* \psi_1 - \psi_2^* \psi_2)}{(\psi_1^* \psi_1 + \psi_2^* \psi_2)}$$

Equations of motion?

Many-particle and mean-field equations of motion

In the hermitian case:

$$\frac{d}{dt}\hat{L}_j = \mathrm{i}[\hat{H}, \hat{L}_j] \quad \rightarrow \quad \frac{d}{dt}l_j = \mathrm{i}\{\mathcal{H}, l_j\}_{\psi_j,\psi_j^*}$$

Now:

$$\begin{aligned} \frac{d}{dt}\langle\Psi|\hat{F}|\Psi\rangle &= \mathrm{i}\langle\Psi|\hat{H}^\dagger\hat{F} - \hat{F}\hat{H}|\Psi\rangle \\ \frac{d}{dt}f &= \mathrm{i}\sum_k \frac{\partial f}{\partial\psi_k^*}\frac{\partial\mathcal{H}^*}{\partial\psi_k} - \frac{\partial f}{\partial\psi_k}\frac{\partial\mathcal{H}}{\partial\psi_k^*} \end{aligned}$$

Non-Hermitian Bloch equations

$$\frac{d}{dt}\frac{\langle\Psi|\widehat{L}_j|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{1}{\langle\Psi|\Psi\rangle}\left(\frac{d}{dt}\langle\Psi|\widehat{L}_j|\Psi\rangle - \frac{\langle\Psi|\widehat{L}_j|\Psi\rangle}{\langle\Psi|\Psi\rangle}\frac{d}{dt}\langle\Psi|\Psi\rangle\right)$$

Yields:

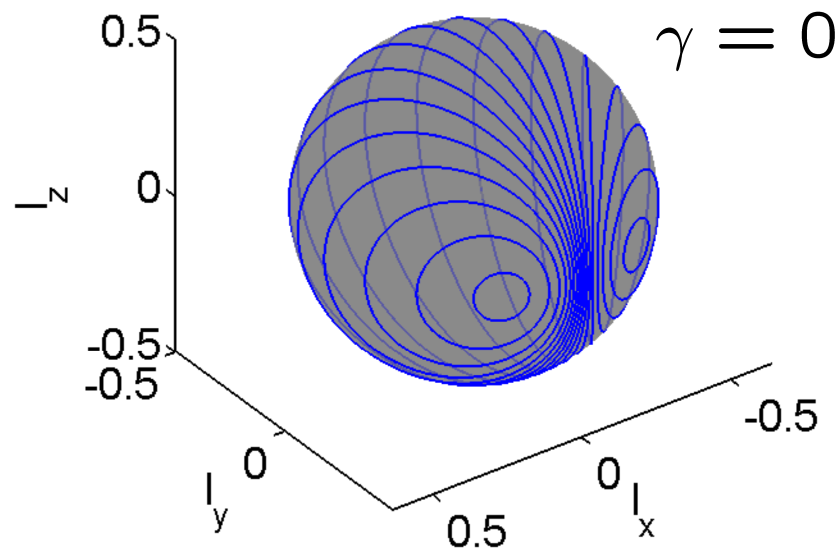
$$\begin{aligned}
\langle\dot{\hat{L}}_x\rangle &= 4\gamma\langle\hat{L}_x\rangle\langle\hat{L}_z\rangle \\
\langle\dot{\hat{L}}_y\rangle &= -2v\langle\hat{L}_z\rangle + 4\gamma\langle\hat{L}_y\rangle\langle\hat{L}_z\rangle \\
\langle\dot{\hat{L}}_z\rangle &= 2v\langle\hat{L}_y\rangle - 4\gamma\left(\langle\hat{L}_x\rangle^2 + \langle\hat{L}_y\rangle^2\right)
\end{aligned}
\qquad
\begin{aligned}
\dot{l}_x &= 4\gamma l_x l_z \\
\dot{l}_y &= -2v l_z + 4\gamma l_y l_z \\
\dot{l}_z &= 2v l_y - 4\gamma(l_x^2 + l_y^2)
\end{aligned}$$

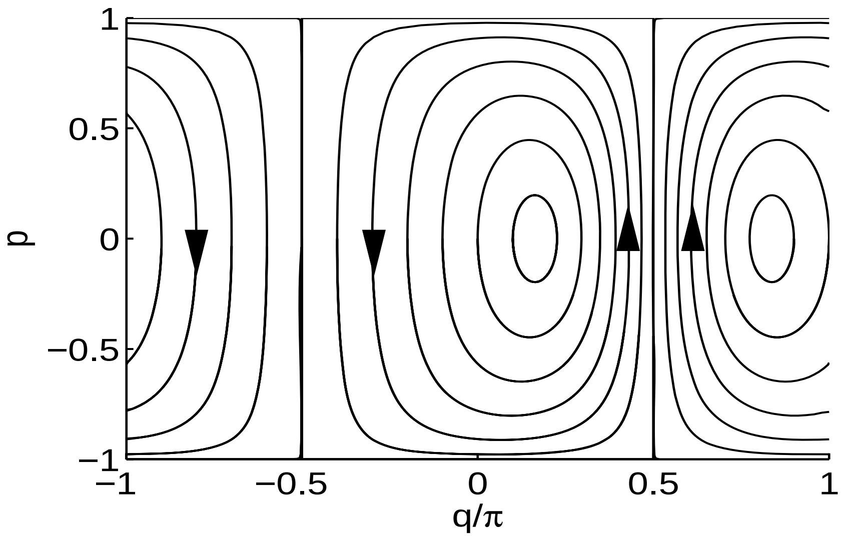
The mean-field approximation is exact if $\mathcal{H}$ is linear.

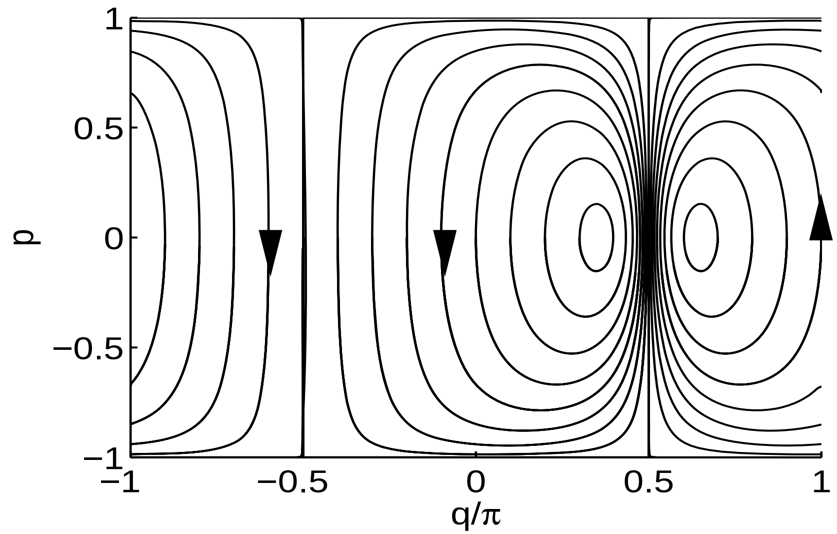
Mean-field dynamics

$\gamma = 0.5v$

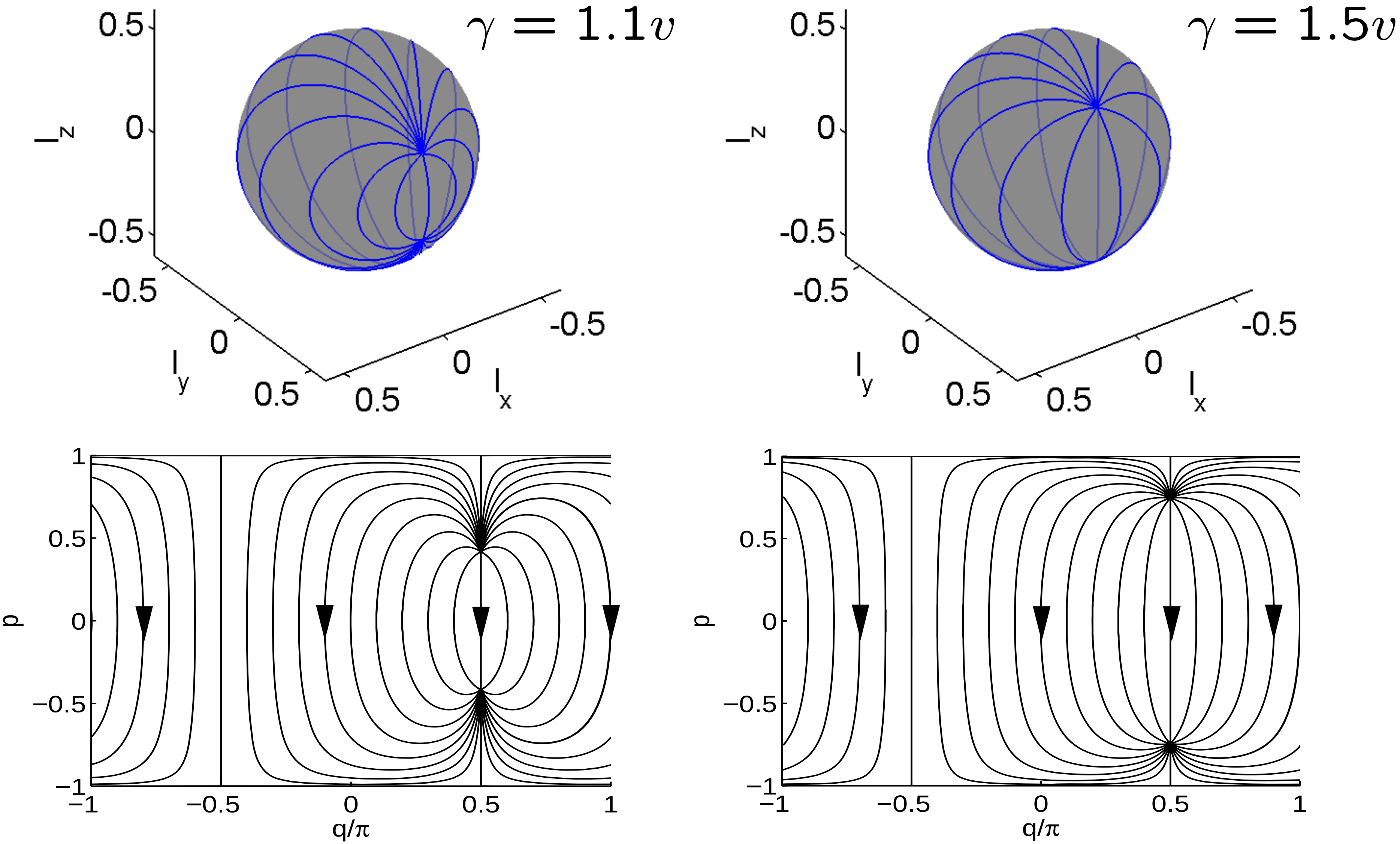
$\gamma = 0.9v$







Mean-field dynamics



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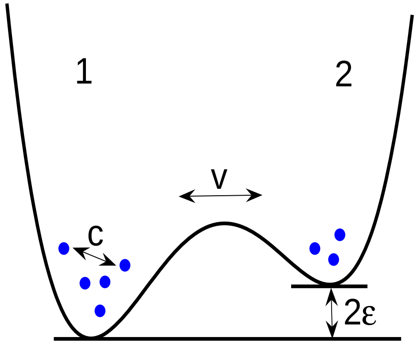

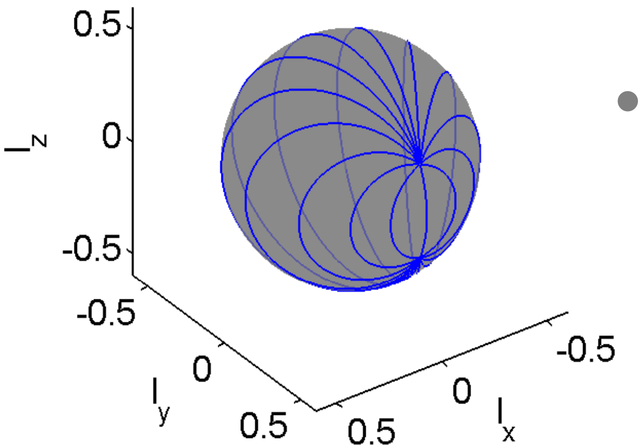

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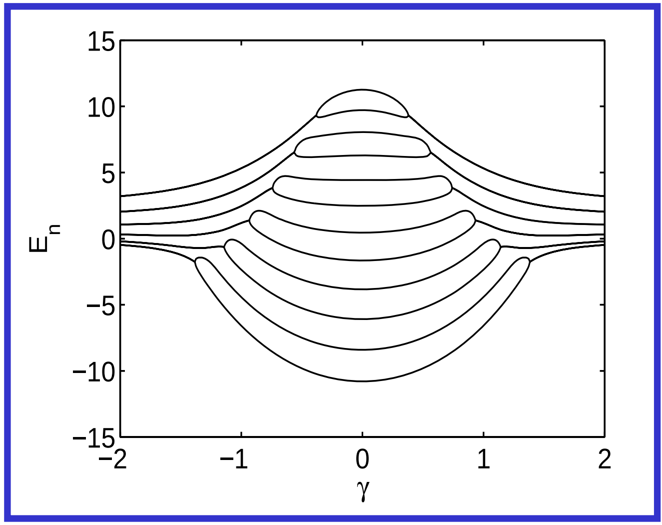
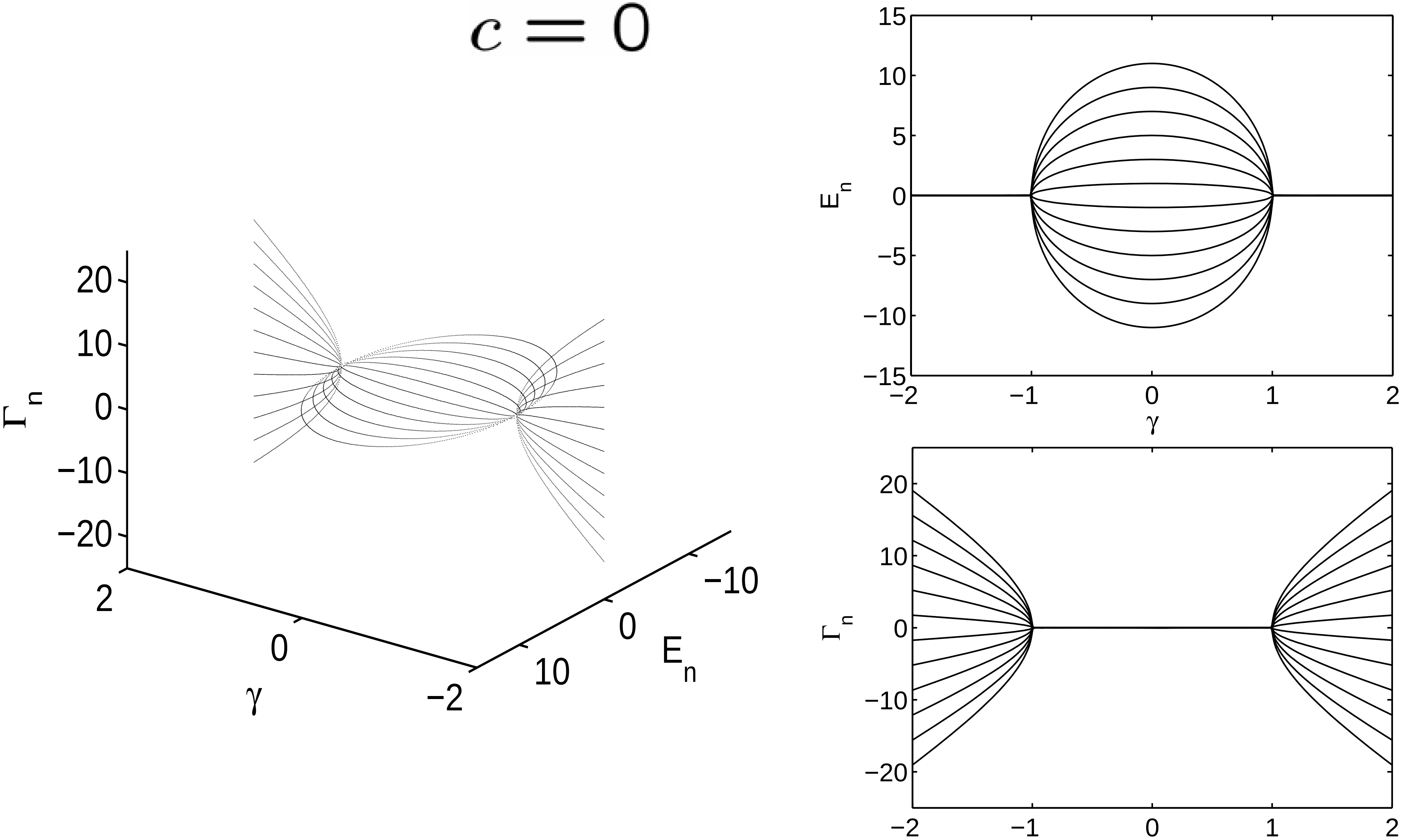


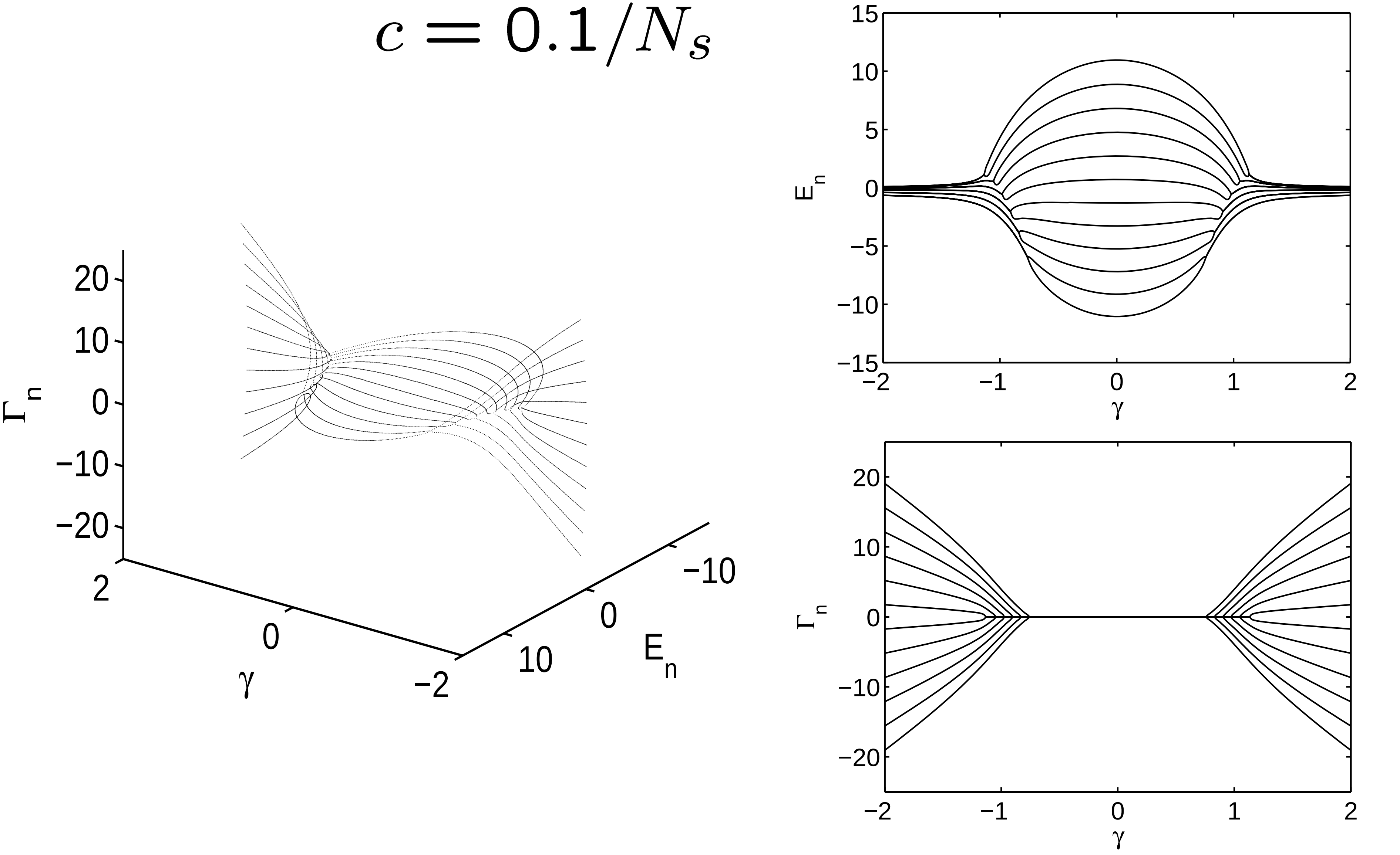
Eigenvalues for N=11 particles

$$c = 0$$



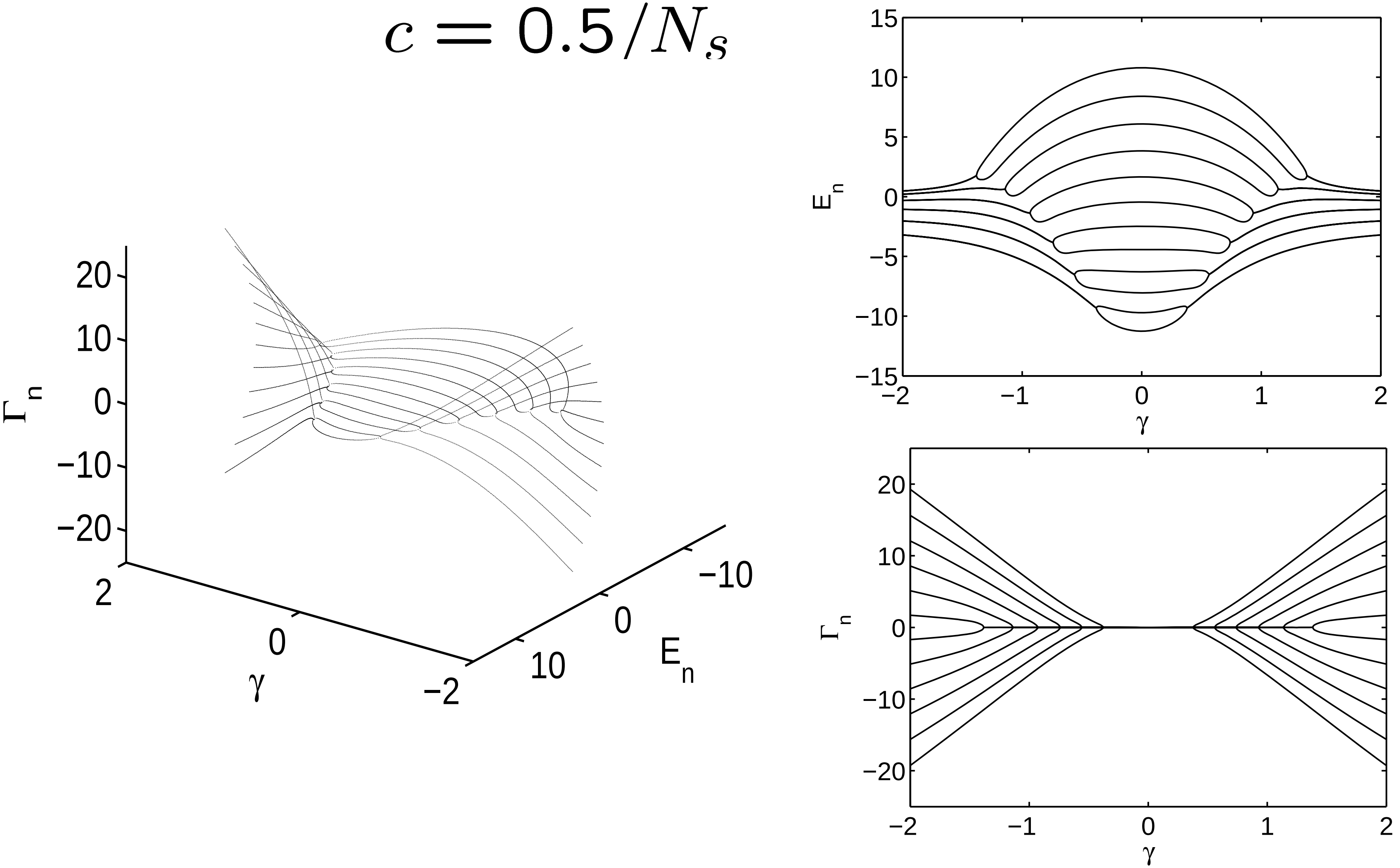
Eigenvalues for N=11 particles

$$c = 0.1/N_s$$



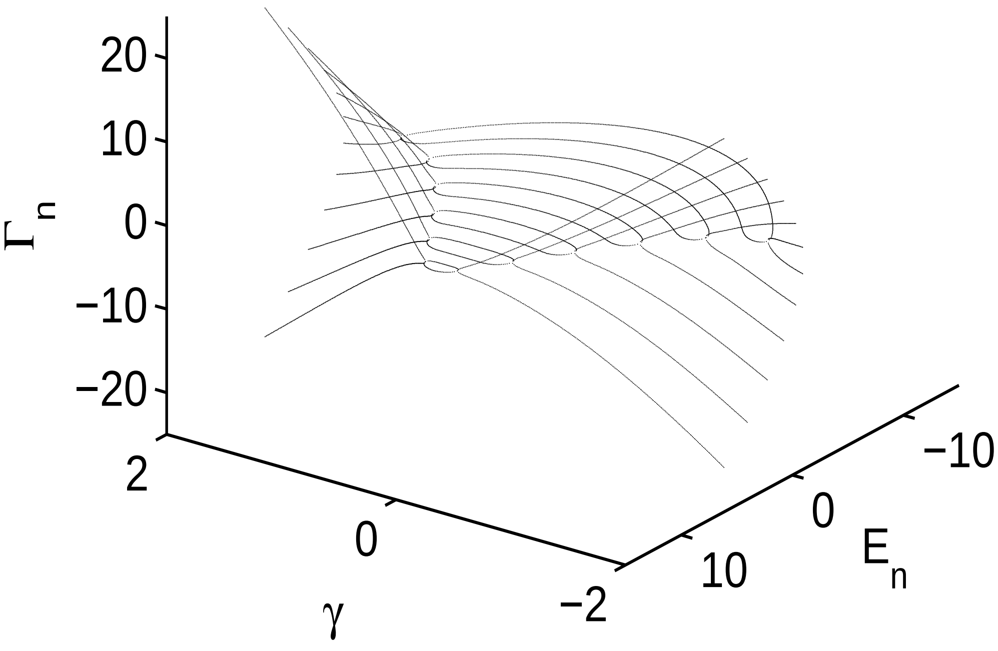
Eigenvalues for N=11 particles

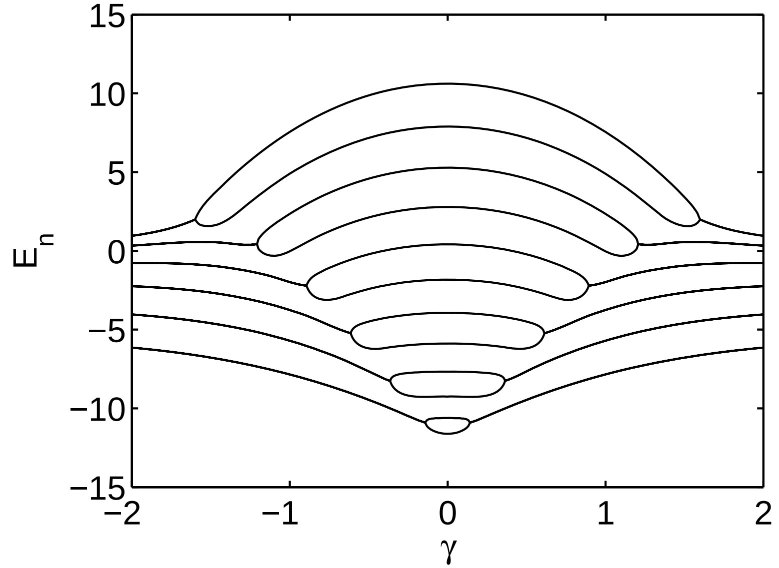
$$c = 0.5/N_s$$

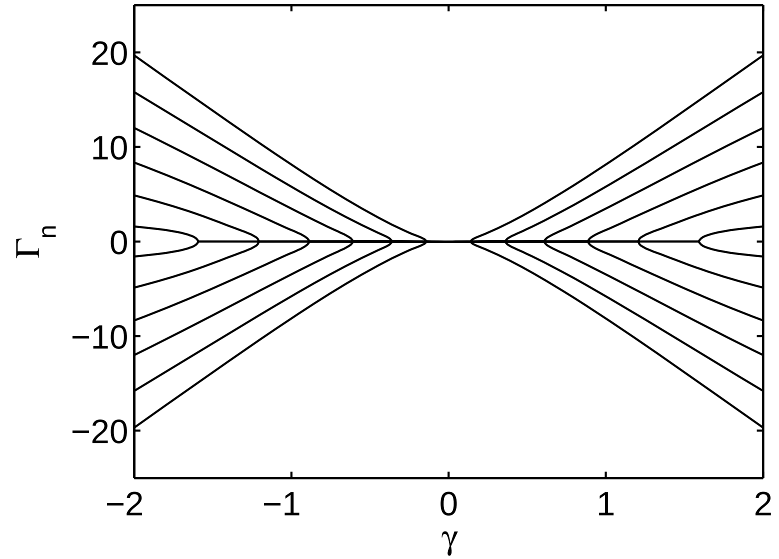


Eigenvalues for N=11 particles

$$c = 1/N_s$$

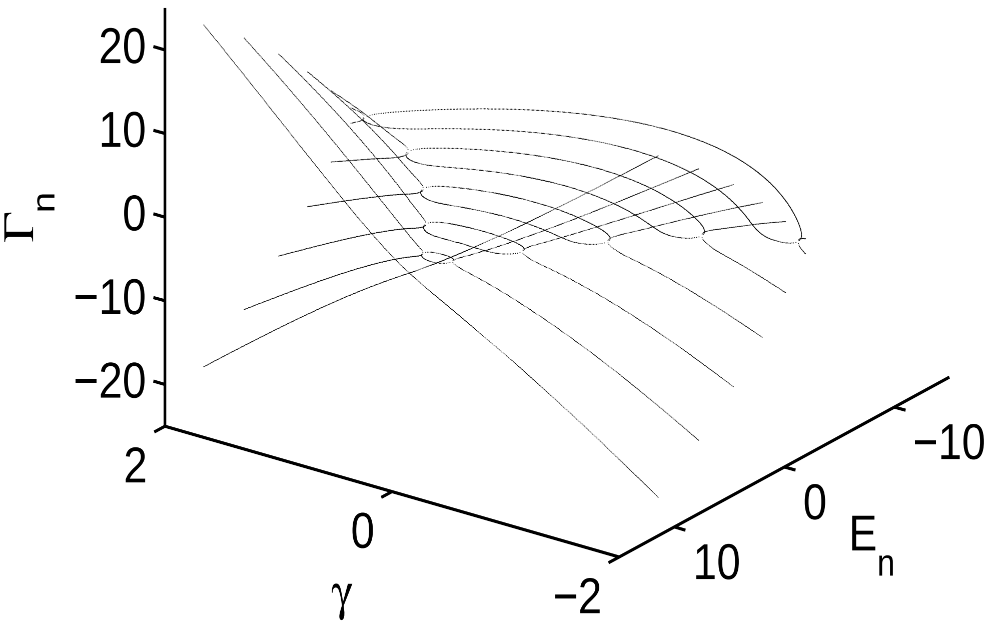


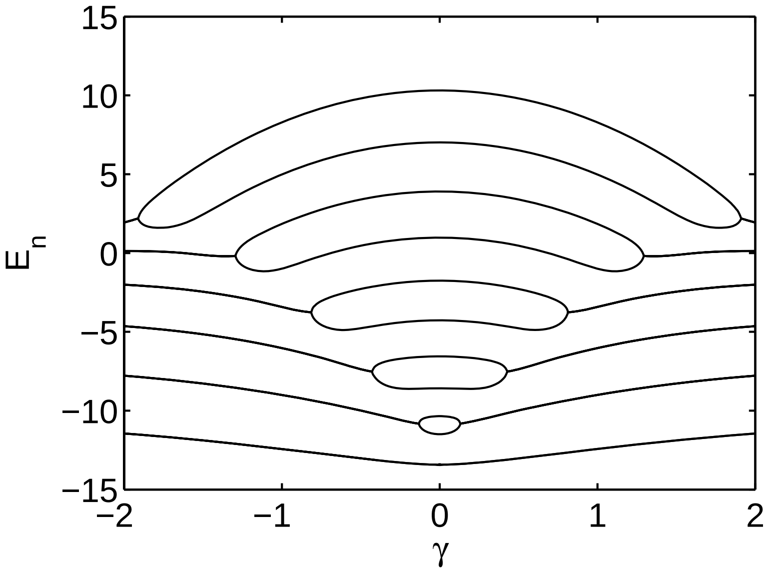


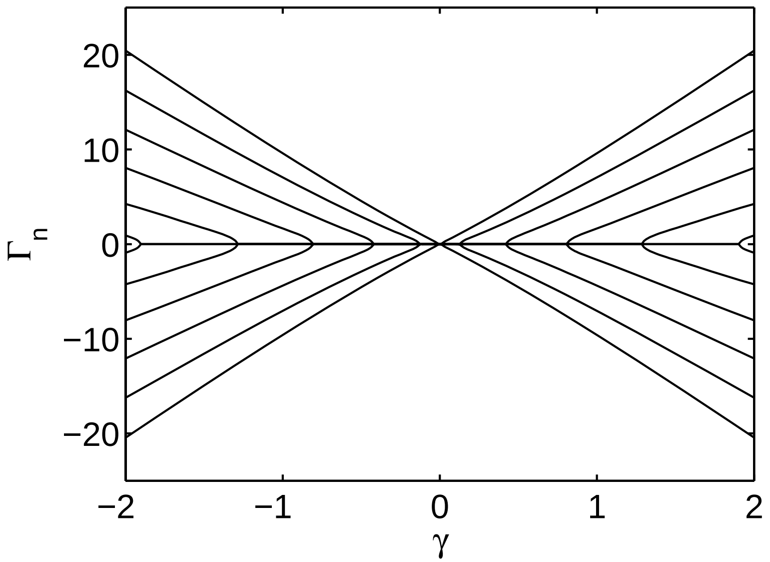


Eigenvalues for N=11 particles

$$c = 2/N_s$$

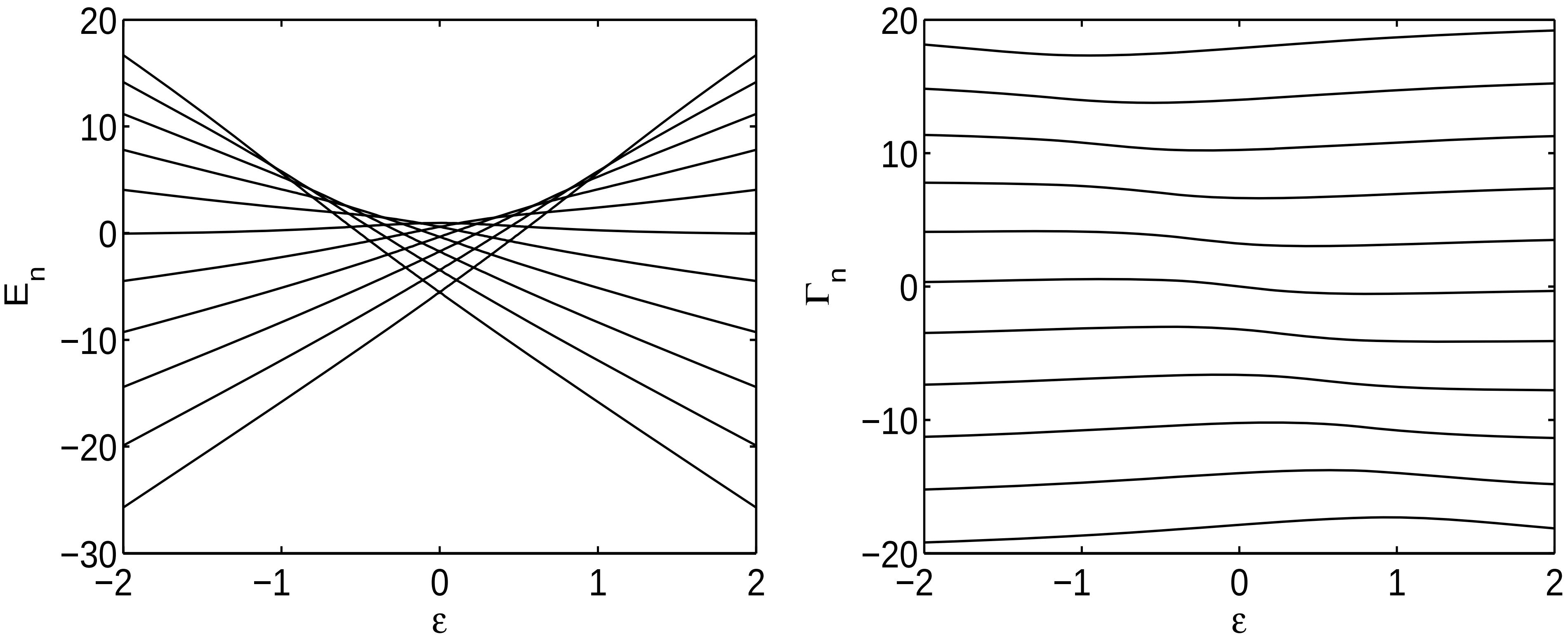






Eigenvalues for the asymmetric case

$$c = 1/N_s \quad \gamma = 2.0$$



Conclusion

- The mean-field system is a classical approximation of the many-particle system.
- Effective non-Hermiticity leads to bifurcations of the resonances.
- The bifurcations are accompanied by symmtery-breaking of the wavefunctions.
- The non-Hermitian mean-field approximation is exact for vanishing interaction.
- Introducing an interaction the resonances bifurcate in pairs.
- Introducing a non-Hermiticity reduces the critical interaction strength.