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A master non-Hermitian cubic *PT*-symmetric Hamiltonian

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Based on:

Paulo Assis and A. Fring, *A master non-Hermitian cubic PT-symmetric Hamiltonian, in preparation*

Outline

- Motivation
  - Reggeon field theory on a lattice
- A master Hamiltonian of cubic order
- Pseudo-Hermitian Hamiltonians from Moyal products
  - generalities
  - construction of the metric
  - uniqueness of the solution
- Quasi-solvable models of cubic order
  - exact solutions for the metric and Hermtian counterparts
- Specific examples
  - Reggeon field theory on a lattice
  - Unbounded potentials
- Conclusions

Motivation

- Non-Hermitian Hamiltonian systems have been studied for some time, resonance systems [von Neumann, Wigner 1929] [Breit-Wigner 1936]
- Here non-Hermitian Hamiltonians with real spectrum, such as

$$H = \frac{1}{2}p^2 + x^2(ix)^\nu \qquad \text{for } \nu \geq 0$$

[C.M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243]

- Since then roughly three phases

i) Spectral analysis: How to explain the reality of the spectrum?

a) Pseudo-hermiticity:

$$H^\dagger = \eta^2 H \eta^{-2} \quad \Leftrightarrow \quad h = \eta H \eta^{-1} = h^\dagger$$

b) PT-symmetry:

$$[H, \mathcal{PT}] = 0 \;\wedge\; \mathcal{PT}\Phi = \Phi \Rightarrow \varepsilon = \varepsilon^* \;\; \text{for } H\Phi = \varepsilon\Phi$$

c) supersymmetry (Darboux transformations)

ii) Formulation of consistent quantum mechanics

iii) Applications (coupling to Laser fields, brachistochrone, etc)

- Non-Hermitian Hamiltonians whose spectra were believed to be real have occurred much earlier in the literature:
  - Lattice version of Reggeon field theory

$$H_{\mathrm{LR}} = \sum_{\vec{i}} \left[\Delta a^\dagger_{\vec{i}} a_{\vec{i}} + i g a^\dagger_{\vec{i}} (a_{\vec{i}} + a^\dagger_{\vec{i}}) a_{\vec{i}} + \tilde{g} \sum_{\vec{j}} (a^\dagger_{\vec{i}+\vec{j}} - a^\dagger_{\vec{i}})(a_{\vec{i}+\vec{j}} - a_{\vec{i}}) \right]$$

[J. Cardy, R. Sugar, Phys. Rev. D12 (1975) 2514]

[R. Brower, M. Furman, K. Subbarao, Phys. Rev. D12 (1977) 1756]

- on a single site:

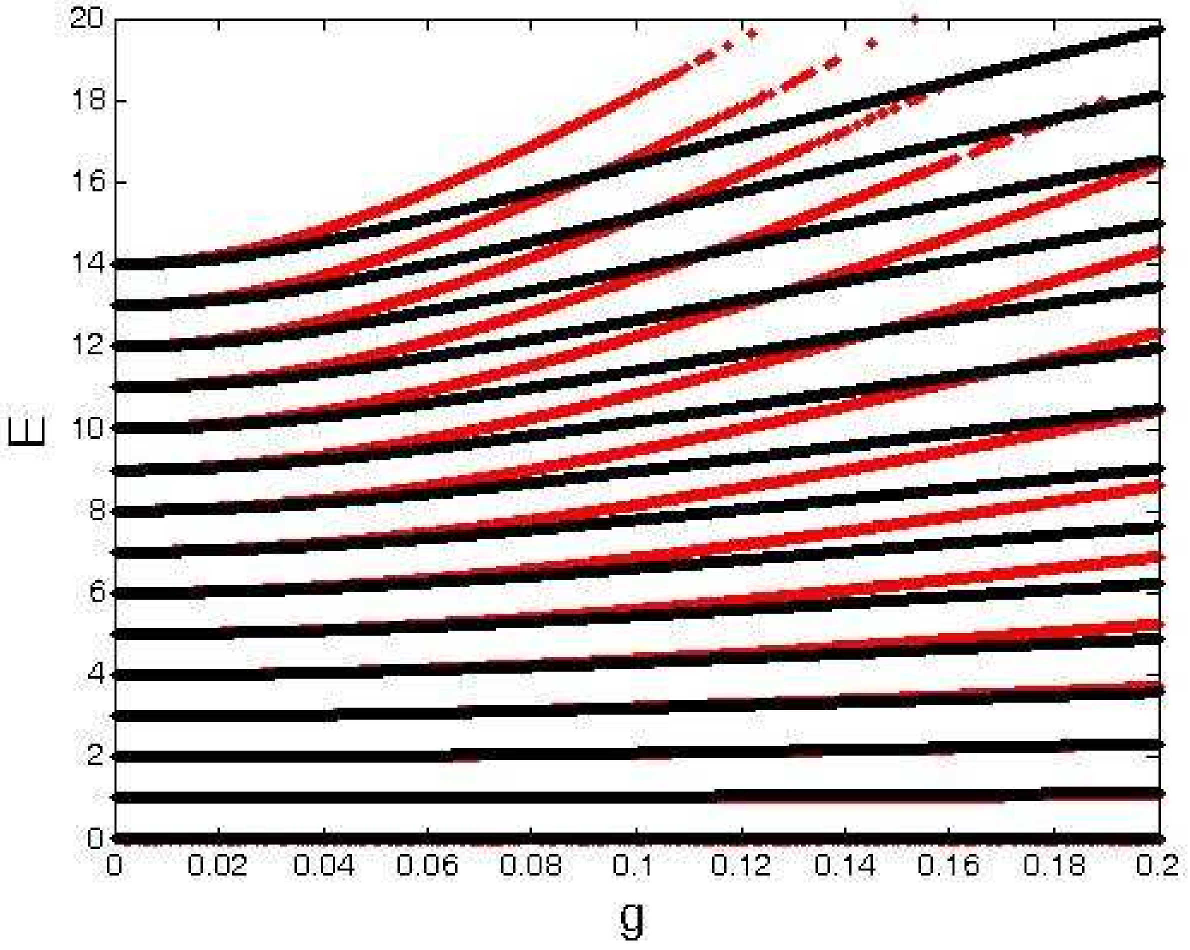
$$H_{\mathrm{SSLR}} = \Delta a^\dagger a + i g a^\dagger (a + a^\dagger) a$$

$$= \frac{\Delta}{2}(\hat{p}^2 + \hat{x}^2 - 1) + i \frac{g}{\sqrt{2}} (\hat{x}^3 + \hat{p}^2 \hat{x} - 2\hat{x} + i\hat{p})$$

with: $a = \frac{1}{\sqrt{2\omega}}(\omega\hat{x} + i\hat{p}), \; a^\dagger = \frac{1}{\sqrt{2\omega}}(\hat{x} - i\omega\hat{p}), \; \omega = 1$

- obviously this Hamiltonian is PT-symmetric:

$\mathcal{PT}: \hat{x} \to -\hat{x}, \hat{p} \to \hat{p}, i \to -i, a \to -a, a^\dagger \to -a^\dagger$ $\Rightarrow$



$$H(\hat{x},\hat{p}) = \frac{1}{2}(\hat{p}^2 + \hat{x}^2 - 1) + i\frac{g}{\sqrt{2}}\hat{x}^3$$

$$H_{\mathrm{SSLR}}(\hat{x},\hat{p}) = \frac{1}{2}(\hat{p}^2 + \hat{x}^2 - 1) + i\frac{g}{\sqrt{2}}(\hat{x}^3 + \hat{p}^2\hat{x} - 2\hat{x} + i\hat{p})$$

- modified programme of [Weidenmüller, J. Phys. A39 (2006) 10229]

A Master Hamiltonian of cubic order

$$H_c = \lambda_1 a^\dagger a + \lambda_2 a^\dagger a^\dagger + \lambda_3 aa + \lambda_4 + i(\lambda_5 a^\dagger + \lambda_6 a + \lambda_7 a^\dagger a^\dagger a^\dagger + \lambda_8 a^\dagger a^\dagger a + \lambda_9 a^\dagger aa + \lambda_{10} aaa)$$

$$H_c = \alpha_1 \hat{p}^3 + \alpha_2 \hat{p}^2 + \alpha_3 \frac{\{\hat{p}, \hat{x}^2\}}{2} + \alpha_4 \hat{p} + \alpha_5 \hat{x}^2 + \alpha_6 + ig\left[\alpha_7 \frac{\{\hat{p}^2, \hat{x}\}}{2} + \alpha_8 \frac{\{\hat{p}, \hat{x}\}}{2} + \alpha_9 \hat{x}^3 + \alpha_{10} \hat{x}\right]$$

- ten parameter family with $\alpha_i, \lambda_i \in \mathbb{R}$, $\alpha = M\lambda$, $\lambda = M^{-1}\alpha$

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{3}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & -\frac{3}{2\sqrt{2}g} \\ 0 & -\frac{1}{g} & \frac{1}{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} & \frac{1}{2\sqrt{2}g} \\ 0 & 0 & 0 & 0 & \frac{1}{g\sqrt{2}} & \frac{1}{g\sqrt{2}} & 0 & -\frac{1}{g\sqrt{2}} & -\frac{1}{g\sqrt{2}} & 0 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{g}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{g}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & \frac{g}{\sqrt{2}} \\ -\frac{3}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & \frac{g}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{g}{2\sqrt{2}} & 0 & \frac{g}{2\sqrt{2}} & 0 \\ \frac{3}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2^{\frac{3}{2}}} & 0 \\ -\frac{3}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & \frac{g}{2\sqrt{2}} & 0 & \frac{3g}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{g}{2\sqrt{2}} & 0 & \frac{g}{2\sqrt{2}} & 0 \end{pmatrix}$$

• for special choices of the parameters this reduces to some well studied models:

model\constants	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
lattice Reggeon	0	$\frac{\Delta}{2}$	0	0	$\frac{\Delta}{2}$	$-\frac{\Delta}{2}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$-\frac{2}{\sqrt{2}}$
ix^3- potential	0	1	0	0	0	0	0	0	1	0
massive ix^3- potential	0	1	0	0	m^2	0	0	0	1	0
$-x^4$-potential	0	1	0	$\frac{1}{2}-g$	α	$-\alpha$	$1/g$	0	0	$-2\alpha/g$
Swanson model	0	$\frac{1}{2}$	0	0	$\frac{\alpha}{2}$	$\frac{g}{2}$	0	1	0	0
ix- potential	0	1	0	0	0	0	0	0	0	1
$-x^6$- potential	0	1	0	$\frac{1}{2}-g$	$3\hat{\alpha}$	$-\hat{\alpha}$	$1/g$	0	$\hat{\alpha}/g$	$-3\hat{\alpha}/g$

• Task: Solve

$$h = \eta H \eta^{-1} = h^\dagger = \eta^{-1} H^\dagger \eta \quad \Leftrightarrow \quad H^\dagger = \eta^2 H \eta^{-2}, \qquad \text{for } \eta = \eta^\dagger$$

for the metric $\eta^2 \Rightarrow \eta \Rightarrow h$

Pseudo Hermitian Hamiltonians from Moyal products

- Exploit isomorphism

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p}) \cong F(x,p)\star G(x,p)$$

  - operator valued functions: $F(\hat{x},\hat{p}),\ G(\hat{x},\hat{p})$
  - scalar functions: $F(x,p),\ G(x,p) \in \mathcal{S}$
  - space of complex integrable functions: $\mathcal{S}$
  - Moyal product: $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$

$$F(x,p) \star G(x,p) = F(x,p) e^{\frac{i}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} G(x,p)$$

$$= \sum_{s=0}^{\infty} \frac{(-i/2)^s}{s!} \sum_{t=0}^{s} (-1)^t \binom{s}{t} \partial_x^t \partial_p^{s-t} F(x,p) \partial_x^{s-t} \partial_p^t G(x,p)$$

  - properties: associative, distributive, Hermiticity: $(F \star G)^* = G^* \star F^*$

  - representation:

$$F(\hat{x},\hat{p}) = \int_{-\infty}^{\infty} ds dt f(s,t) e^{i(s\hat{x}+t\hat{p})}$$

$$F(x,p) = \int_{-\infty}^{\infty} ds dt f(s,t) e^{i(sx+tp)}$$

- compute:

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p}) = \int_{-\infty}^{\infty} ds dt ds' dt' f(s,t) f(s',t') e^{\frac{i}{2}(ts'-t's)} e^{i(s+s')\hat{x}+i(t+t')\hat{p}}$$

⇒ the isomorphism is guaranteed by the definition above

- Hermiticity: $F^\dagger(\hat{x},\hat{p}) = F(\hat{x},\hat{p}) \cong F^*(x,p) = F(x,p)$
- In this context Moyal products have been employed before in

[F. Scholtz, H. Geyer, Phys. Lett. B634 (2006) 84]

[F. Scholtz, H. Geyer, J. Phys. Lett. A39 (2006) 10189]

[C. Figueira de Morisson Faria, A. Fring, Czech. J. Phys. 56 (2006) 899]

- SG use asymmetrical definition:

$$F(x,p) * G(x,p) = F(x,p) e^{i \overleftarrow{\partial}_x \overrightarrow{\partial}_p} G(x,p)$$

⇒ i) $(F * G)^* \neq G^* * F^*$

ii) $F^\dagger(\hat{x},\hat{p}) = F(\hat{x},\hat{p}) \cong F^*(x,p) = e^{-i\partial_x\partial_p} F(x,p)$

iii) the differential equation are more complicated

- Construction of $\eta^2(\hat{x},\hat{p}),\ \eta(\hat{x},\hat{p}),\ h(\hat{x},\hat{p})$:
  - $H(\hat{x},\hat{p}) \to H(x,p)$
  - Solve differential equation for $\eta^2(x,p)$

$$H^\dagger(x,p)\star\eta^2(x,p) = \eta^2(x,p)\star H(x,p)$$

$$\cong H^\dagger(\hat{x},\hat{p})\eta^2(\hat{x},\hat{p}) = \eta^2(\hat{x},\hat{p})H(\hat{x},\hat{p})$$

  - Solve $\eta^2(x,p) = \eta(x,p)\star\eta(x,p)$ for $\eta(x,p)$

$$\cong \eta^2(\hat{x},\hat{p}) = \eta(\hat{x},\hat{p})\eta(\hat{x},\hat{p})$$

  - Compute $\eta^{-1}(x,p)$
  - Compute $h(x,p) = \eta(x,p)\star H(x,p)\star\eta^{-1}(x,p)$
  - Check Hermiticity:

$$F^\dagger(\hat{x},\hat{p}) = F(\hat{x},\hat{p}) \cong F^*(x,p) = F(x,p)$$

  - positive definiteness: $\log F(x,p) \in \mathbb{R}$
  - $\eta^2(x,p),\ \eta(x,p),\ h(x,p) \to \eta^2(\hat{x},\hat{p}),\ \eta(\hat{x},\hat{p}),\ h(\hat{x},\hat{p})$

➤ Ambiguities:

Obviously taking H as the only starting point, there is not one unique Hermitian counterpart h in the same similarity class.

- different types of mechanisms:
  - If η^2 solves $H^\dagger = \eta^2 H \eta^{-2}$ then

$$\hat{\eta}^2 = (H^\dagger)^n \eta^2 H^n \qquad \text{for } n \in \mathbb{N}$$

is also a solution.

  - perturbative ansatz: $\eta^2 = e^{gq}$

    at each order $q \to q + h_0$ is an ambiguity
  - any symmetry is an ambiguity $\eta^2 \to \eta^2 Q$ with $[Q, H] = 0$
- fix by:
  - eliminating non-physical solutions
  - choose a further observable

    [F.Scholtz, H. Geyer, F. Hahne, Ann. Phys. 213 (1992) 74]

    but this is artificial as one does not know beforehand which to choose

Quasi-solvable models of cubic order

- Convert $H_c(\hat{x},\hat{p}) \to H_c(x,p)$

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6$$
$$+ ig(\alpha_7 p^2 x + \alpha_8 p x + \alpha_9 x^3 + \alpha_{10} x)$$

$$H^\dagger(x,p) \star \eta^2(x,p) = \eta^2(x,p) \star H(x,p)$$

⇒ differential equation

$$\left(\alpha_3 p x \partial_p + \alpha_5 x \partial_p + \frac{\alpha_3}{8}\partial_x \partial_p^2 + \frac{\alpha_1}{8}\partial_x^3 - \alpha_2 p \partial_x - \frac{3}{2}\alpha_1 p^2 \partial_x - \frac{\alpha_3}{2} x^2 \partial_x - \frac{\alpha_4}{2}\partial_x\right)\eta^2$$
$$= g\left(\alpha_9 x^3 + \alpha_{10} x + \alpha_8 p x + \alpha_7 p^2 x + \frac{\alpha_7}{2} p \partial_x \partial_p + \frac{\alpha_8}{4}\partial_x \partial_p - \frac{\alpha_7}{4} x \partial_x^2 - \frac{3}{4}\alpha_9 x \partial_p^2\right)\eta^2$$

- Ansatz for solution:

$$\eta^2(x,p) = \exp g(q_1 p^3 + q_2 p x^2 + q_3 p^2 + q_4 x^2 + q_5 p)$$

- assume perturbative expansion in g
- solve systemmatically order by order to construct exact solutions

➤ Solutions are governed by various constraints on the α:

Non vanishing $\hat{p}\hat{x}^2$-term ($\alpha_3 \neq 0$)

- with additional constraints

$$\alpha_1 = \alpha_7 = 0, \ \alpha_3\alpha_{10} = \alpha_4\alpha_9, \ \alpha_3\alpha_8 = 2\alpha_2\alpha_9$$

$$H_c(x,p) = \alpha_2 p^2 + \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig\left(\frac{2\alpha_2\alpha_9}{\alpha_3} p x + \alpha_9 x^3 + \frac{\alpha_4\alpha_9}{\alpha_3} x\right)$$

- the differential equation is solved exactly by

$$\eta^2(x,p) = e^{-g\frac{\alpha_9}{\alpha_3}x^2}$$

$$\Rightarrow h_c(x,p) = \alpha_2 p^2 + \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\alpha_2 \alpha_9^2}{\alpha_3{}^2} x^2$$

- the metric is Hermitian and positive definite
- solution does not reduce to any of the well studied models

Non vanishing $\hat{p}\hat{x}^2$-term ($\alpha_3 \neq 0$)

- with additional constraints

$$\alpha_1\alpha_9 = \alpha_3\alpha_7, \quad \alpha_2\alpha_9 = \alpha_5\alpha_7 + \alpha_3\alpha_8, \quad \alpha_4\alpha_9 = \alpha_5\alpha_8 + \alpha_3\alpha_{10}$$

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_3 p x^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6$$
$$+ig\left(\frac{\alpha_1\alpha_9}{\alpha_3} p^2 x + \frac{\alpha_2\alpha_9 - \alpha_5\alpha_7}{\alpha_3} p x + \alpha_9 x^3 + \frac{\alpha_4\alpha_9 - \alpha_5\alpha_8}{\alpha_3} x\right)$$

- the differential equation can be solved exactly

$$\eta^2(x,p) = e^{-g\left(\frac{\alpha_7}{\alpha_3} p^2 + \frac{\alpha_8}{\alpha_3} p + \frac{\alpha_9}{\alpha_3} x^2\right)}$$

- however $\eta(x,p) \star \eta(x,p) = \eta^2(x,p)$ is solved perturbatively

$$\eta(x,p) = 1 - g\frac{\alpha_7 p^2 + \alpha_8 p + x^2\alpha_9}{2\alpha_3} + g^2\left(\frac{(p\alpha_7 + \alpha_8)^2 p^2 + \alpha_9^2 x^4}{8\alpha_3^2}\right.$$
$$\left. + \frac{\alpha_9(\alpha_7 + 2\alpha_7 p^2 x^2 + 2\alpha_8 p x^2)}{8\alpha_3^2}\right) + \mathcal{O}(g^3)$$

$$\Rightarrow \quad h_c(x,p) = \alpha_3 p x^2 + \alpha_5 x^2 + \alpha_6 + \frac{\alpha_3\alpha_7}{\alpha_9} p^3 + \frac{(\alpha_5\alpha_7 + \alpha_3\alpha_8)}{\alpha_9} p^2 + \frac{(\alpha_5\alpha_8 + \alpha_3\alpha_{10})}{\alpha_9} p$$

$$- g^2 \frac{(2\alpha_7 p + \alpha_8)(p(\alpha_7 p + \alpha_8) + \alpha_9 x^2 + \alpha_{10})}{4\alpha_3} + \mathcal{O}(g^4)$$

- the metric is Hermitian and positive definite
- solution does not reduce to any of the well studied models

Non vanishing $\hat{p}\hat{x}^2$-term and vanishing $\hat{x}^3$-term

- with only one additional constraint

$$\alpha_{10}\alpha_3^2 = \alpha_5(\alpha_3\alpha_8 - \alpha_5\alpha_7)$$

$$H_c(x,p) = h_0(x,p) + ig(\alpha_7 p^2 x + \alpha_8 p x + \frac{\alpha_5(\alpha_3\alpha_8 - \alpha_5\alpha_7)}{\alpha_3^2} x)$$

- the differential equation is solved exactly by

$$\eta^2(x,p) = \eta^2(p) = e^{g\left(\frac{\alpha_7}{2\alpha_3} p^2 + \frac{\alpha_3\alpha_8 - \alpha_5\alpha_7}{\alpha_3{}^2} p\right)}$$

$$\Rightarrow \quad h_c = h_0 + g^2\left(\frac{\alpha_7{}^2}{4\alpha_3} p^3 + \frac{2\alpha_3\alpha_7\alpha_8 - \alpha_5\alpha_7^2}{4\alpha_3{}^2} p^2 + \frac{\alpha_3{}^2\alpha_8{}^2 - \alpha_5{}^2\alpha_7{}^2}{4\alpha_3{}^3} p + \frac{\alpha_5(\alpha_5\alpha_7 - \alpha_3\alpha_8)^2}{4\alpha_3{}^4}\right)$$

- the metric is Hermitian and positive definite
- solution does not reduce to any of the well studied models
- relax the constraint by demanding a different form of the solution

$$\eta^2(p) = (p\alpha_3 + \alpha_5)^{\frac{g(\alpha_5^2\alpha_7 - \alpha_3\alpha_5\alpha_8 + \alpha_3^2\alpha_{10})}{\alpha_3^3}} e^{g\left(\frac{\alpha_7}{2\alpha_3}p^2 + \frac{\alpha_3\alpha_8 - \alpha_5\alpha_7}{\alpha_3^2}p\right)}$$

$$\Rightarrow \quad h_c(x,p) = h_0 + g^2 \frac{(p^2\alpha_7 + p\alpha_8 + \alpha_{10})^2}{4(p\alpha_3 + \alpha_5)}$$

Vanishing $\hat{p}\hat{x}^2$-term and non vanishing $\hat{x}^2$-term

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_7 p^2 x + \alpha_8 px + \alpha_{10} x)$$

$$\eta^2(x,p) = e^{g\left(\frac{\alpha_7}{3\alpha_5}p^3 + \frac{\alpha_8}{2\alpha_5}p^2 + \frac{\alpha_{10}}{\alpha_5}p\right)}$$

$$h_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{(p^2\alpha_7 + p\alpha_8 + \alpha_{10})^2}{4\alpha_5}$$

- the metric is Hermitian and positive definite
- reduces to Swanson and transformed $-x^4$ potential of [Jones, Mateo]

Vanishing $\hat{p}\hat{x}^2$-term

$$H_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_8 px + \alpha_{10} x)$$

$$\alpha_1 = \alpha_7 = \alpha_9 = \alpha_3 = 0 \qquad \text{and} \qquad \alpha_4\alpha_8 = 2\alpha_2\alpha_{10}$$

$$\eta^2(x,p) = e^{-g\alpha_{10}/\alpha_4 x^2} \qquad \text{for } \alpha_4 \neq 0$$

$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\alpha_2\alpha_{10}^2}{\alpha_4^2} x^2 \qquad \text{for } \alpha_4 \neq 0$$

$$\eta^2(x,p) = e^{-g\alpha_8/2\alpha_2 x^2} \qquad \text{for } \alpha_2 \neq 0$$

$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2 \frac{\alpha_8^2}{4\alpha_2} x^2 \qquad \text{for } \alpha_2 \neq 0$$

- the metric is Hermitian and positive definite
- case 1 does not reduce to any of the studied models
- case 2 reduces to the Swanson Hamiltonian

These are all exact solutions of the specified form.

Lattice version of Reggeon field theory

$$H_{\text{SSLR}} = \Delta a^\dagger a + iga^\dagger(a + a^\dagger)a$$

- pseudo-Hermiticity

$$H_{\text{SSLR}}^\dagger = \eta^2 H_{\text{SSLR}} \eta^{-2}$$

- parity operator [M. Moshe, Phys. Rep. 37 (1978) 255]

$$\mathcal{P} = e^{\frac{i\pi}{2}a^\dagger a} = \eta^2 \qquad \because \mathcal{P}a\mathcal{P} = -a \quad \mathcal{P}a^\dagger\mathcal{P} = -a^\dagger$$

but $\eta = \sqrt{\mathcal{P}} \neq \eta^\dagger \not\Rightarrow h = h^\dagger$

- alternatively

$$\eta^2 = e^{\frac{i\pi}{2}(aa - a^\dagger a^\dagger)} \qquad \because \quad \eta a \eta^{-1} = ia^\dagger \quad \eta a^\dagger \eta^{-1} = ia$$

$$h = \eta H \eta^{-1} = -\Delta a a^\dagger + ga(a + a^\dagger)a^\dagger$$

but the metric is not positive definite

- this case is not one of the exactly solvable ones ⇒ perturbation theory

Consider: $g \to \tilde{g}\sqrt{2}\,, \Delta = 1$

$$H_{\mathrm{SSLR}}(x,p) = a^\dagger \star a + iga^\dagger \star (a + a^\dagger) \star a$$

$$= \frac{1}{2}(x^2 + p^2 - 1) + i\tilde{g}(\mathrm{x}^3 + p^2 x - 2x)$$

$$(2x\partial_p - 2p\partial_x)\eta^2(x,p) = \tilde{g}(4x^3 - 8x + 4p^2 x + 2p\partial_x\partial_p - 3x\partial_p^2 - x\partial_x^2)\eta^2(x,p)$$

Ansatz: $\eta^2(x,p) = 2\sum_{n=0}^{\infty} \tilde{g}^n c_n(x,p)$

$$(2x\partial_p - 2p\partial_x)c_n(x,p) = (4x^3 - 8x + 4p^2 x + 2p\partial_x\partial_p - 3x\partial_p^2 - x\partial_x^2)c_{n-1}(x,p)$$

$$\lim_{g\to 0} \eta^2(x,p) = 0 \Rightarrow c_0(x,p) = 1$$

$$c_1(x,p) = p^3 - 2p + px^2$$

$$c_2(x,p) = p^6 - 4p^4 + p^2 + x^2 - 4p^2x^2 + 2p^4x^2 + p^2x^4$$

$$c_3(x,p) = \frac{2}{3}p^9 - 4p^7 - 5p^5 + 24p^3 - 4p + 8px^2 - 6p^3x^2 - 8p^5x^2 + 2p^7x^2$$

$$- px^4 + \frac{2}{3}p^3x^6 + 2p^7x^2 - px^4 + \frac{2}{3}p^3x^6 - 4p^3x^4 + 2p^5x^4$$

Ambiguity: $c_n(x,p) \to c_n(x,p) + \frac{1}{2}(x^2 + p^2 - 1)$

Fix with: $\eta^2(x,p,g) \star \eta^2(x,p,-g) = 1$

Solve: $\eta(x,p) \star \eta(x,p) = \eta^2(x,p)$

Ansatz: $\eta(x,p) = 1 + \sum_{n=1}^{\infty} \tilde{g}^n q_n(x,p)$

$$q_1(x,p) = c_1(x,p)$$

$$q_2(x,p) = c_2(x,p)/2$$

$$q_3(x,p) = \frac{1}{6}p^9 - p^7 - \frac{17}{4}p^5 + 16p^3 - 3p - \frac{15}{2}p^3x^2 + \frac{1}{2}p^7x^2 + \frac{p^5x^4}{2} + \frac{p^3x^6}{6} - \frac{13}{4}px^4 - p^3x^4 + 12px^2 - 2p^5x^2$$

Hermitian counterpart:

$$h_{\mathrm{SSLR}}(x,p) = \frac{1}{2}(x^2 + p^2 - 1) + g^2\left(\frac{3}{2}p^4 - 4p^2 + 1 - 4x^2 + 3p^2x^2 + \frac{3}{2}x^4\right)$$

$$-g^4\left(\frac{17}{2}p^6 - 34p^4 + 4p^2 + 8 + 4x^2 - 48p^2x^2 + \frac{41}{2}p^4x^2 - 14x^4 + \frac{31}{2}p^2x^4 + \frac{7}{2}x^6\right) + \mathcal{O}(g^6)$$

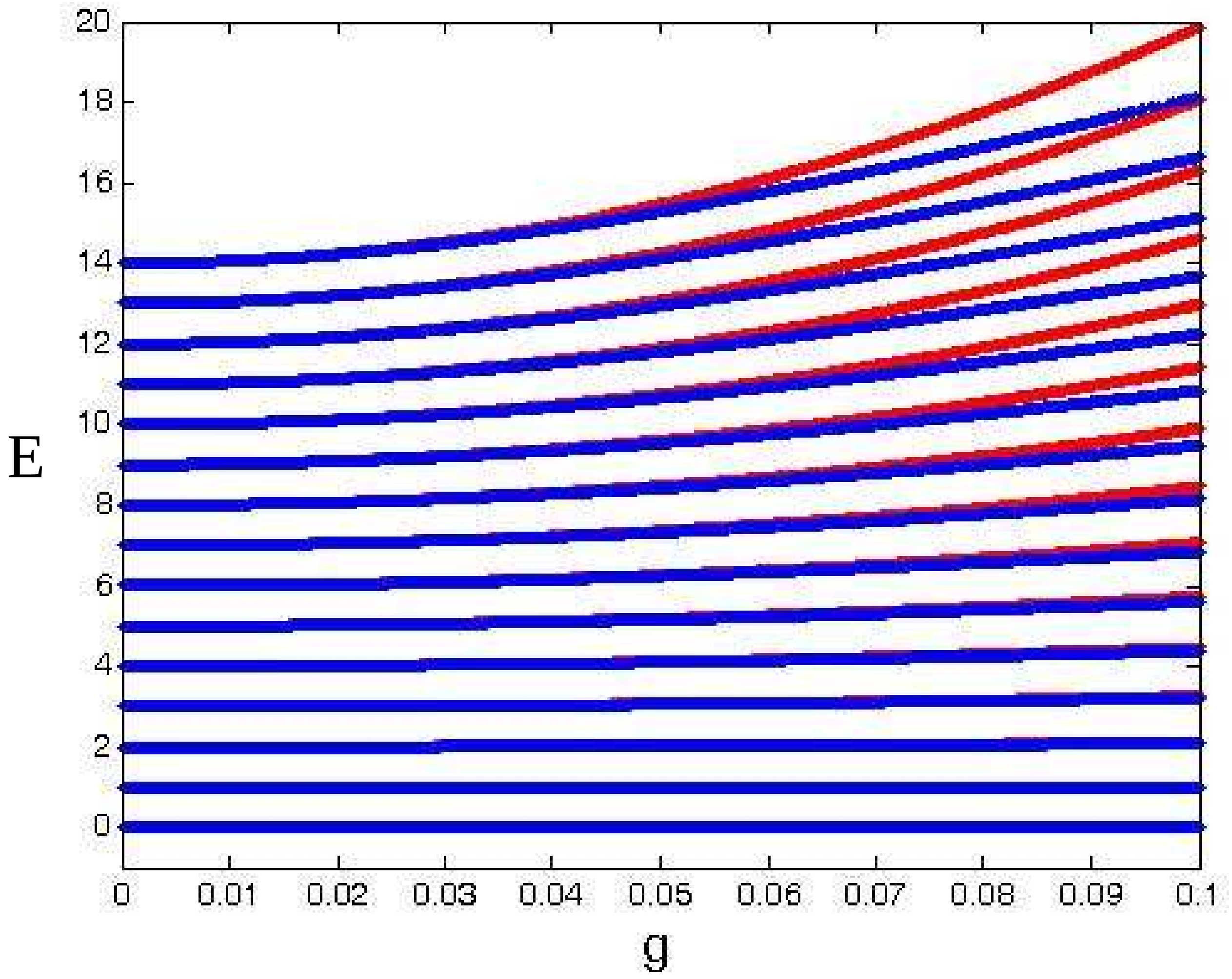
or in terms of creation and annihilation operators:

$$\eta = 1 + i\sqrt{2}\,ga^\dagger(a^\dagger - a)a + \tilde{g}^2a^\dagger[a^\dagger(2a^\dagger a - a^\dagger a^\dagger - aa + 5)a - 2a^\dagger a^\dagger - 2aa + 2]a$$

$$h_{\mathrm{SSLR}} = a^\dagger a + g^2a^\dagger(6a^\dagger a + 4)a + g^4[a^\dagger a^\dagger(10a^\dagger a^\dagger + 10aa - 48a^\dagger a)aa$$

$$+ a^\dagger(20a^\dagger a^\dagger + 20aa - 120a^\dagger a)a - 32a^\dagger a\Big] + \mathcal{O}(g^6)$$

- Numerial solution with square exponential fall off at infinity versus perturbative solution :



---- numerical solution

---- perturbative solution to second order

Unbounded potentials

- transform from wedge with physical decay at infinity to the real axis:

- $$H_4(z) = p_z^2 \pm \varepsilon z^4 \qquad \alpha = 16\varepsilon$$

$$\Rightarrow H_4(\hat{x},\hat{p}) = \hat{p}^2 + \frac{\hat{p}}{2} + i\hat{x}\hat{p}^2 \pm (\alpha + 2i\alpha\hat{x} - \alpha\hat{x}^2)$$

with $z = -2i\sqrt{1+ix}$ [H. Jones, J. Mateo, Phys. Rev. D73 (2006) 085002

- $$H_6(z) = p_z^2 \pm \varepsilon z^6 \qquad \hat{\alpha} = 64\varepsilon$$

  - exponential fall off in

$$\mathcal{W}_L = \left\{\theta \middle| -\tfrac{7}{8}\pi < \theta < -\tfrac{5}{8}\pi\right\} \qquad \mathcal{W}_R = \left\{\theta \middle| -\tfrac{3}{8}\pi < \theta < \tfrac{1}{8}\pi\right\}$$

  - same parameterization

$$H_6(\hat{x},\hat{p}) = \hat{p}^2 + \frac{\hat{p}}{2} + i\hat{x}\hat{p}^2 \pm (\hat{\alpha} + 3i\hat{\alpha}\hat{x} - 3\hat{\alpha}\hat{x}^2 - i\hat{\alpha}\hat{x}^3)$$

  - not one of the quasi-solvable models

Conclusions and further problems

- We classified exact solutions for the most general cubic *PT*-symmetric Hamiltonian.
- The symmetrical definition of the Moyal product is a very useful technical tool to construct the metric operator and Hermitian counterpart for pseudo Hermitian Hamiltonians.

Further problems:

⇒ Quartic potentials

⇒ Lie algebraic generalisations

⇒ higher order potentials, generalisation of the Regge lattice models

$$H = \Delta a^{\dagger} a + \frac{ig}{2} a^{\dagger} (a + a^{\dagger})^{N} a$$

...

Thank you very much for your attention.