PT Symmetry on the Lattice: The Quantum Group Invariant XXZ Spin-Chain

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Why XXZ?

- QM system with finite-dim, PT symmetric, non-Hermitian Hamiltonian H
- Diagonalisable with real eigenvalues (after reduction) $\implies$ quasi-Hermitian
- Integrable, much studied, with several algebraic symmetries
- Can we find simple algebraic expression for η, where

$$\eta H = H^* \eta, \quad \eta \text{ Hermitian, invertible, +ve?}$$

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The quantum group invariant XXZ model

$$H : \overbrace{V \otimes V \otimes \cdots \otimes V}^{N} \to \overbrace{V \otimes V \otimes \cdots \otimes V}^{N}, \quad V = \mathbb{C}^2$$

$$H = \frac{1}{2} \sum_{i=1}^{N-1} \{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta_+ (\sigma_i^z \sigma_{i+1}^z - 1) \} + \Delta_- \frac{(\sigma_1^z - \sigma_N^z)}{2}$$

where $\quad \Delta_\pm = \dfrac{q \pm q^{-1}}{2}$

Can be rewritten as

$$H = \sum_{i=1}^{N-1} E_i, \quad \text{where} \quad E_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & \end{pmatrix}$$

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If $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ then standard inner product defined by

$$\langle v_{s_1} \otimes v_{s_2} \otimes v_{s_N}, v_{s_1'} \otimes v_{s_2'} \otimes v_{s_N'} \rangle = \delta_{s_1,s_1'} \cdots \delta_{s_N,s_N'}.$$

Whence

$$E_i^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\bar{q}^{-1} & 1 & 0 \\ 0 & 1 & -\bar{q} & 0 \\ 0 & 0 & 0 & \end{pmatrix}$$

$q \in \mathbb{R} \implies H = H^*$, $\quad q \in \mathbb{S}^1 \implies H \neq H^*$ [from now on]

P and T defined to act in obvious way as

$$\begin{aligned} P\, v_{s_1} \otimes v_{s_2} \otimes v_{s_N} &= v_{s_N} \otimes v_{s_{N-1}} \otimes v_{s_1} \\ T\,\lambda(v_{s_1} \otimes v_{s_2} \otimes v_{s_N}) &= \bar{\lambda}(v_{s_1} \otimes v_{s_2} \otimes v_{s_N}) \end{aligned}$$

Then we have

$$PE_i = E_{N-i}^* P, \quad TE_i = E_i^* T \implies [PT, H] = 0$$

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$[PT, H] = 0 \implies$ all eigenvalues real or come in complex conjugate pairs.

Known properties

- When $q \in \mathbb{S}^1$ not a root of unity, H diagonalisable with real spectrum.
- When q a root of unity, non-trivial Jordans blocks, but after a suitable reduction of space, diagonalisable with real spectrum.

Finite dim, diagonalisable H with real spectrum is *quasi-Hermitian*, i.e., there exists η, Hermitian, invertible, $+$ve, such that $\eta H = H^* \eta$.

Such an η defines new inner product $\langle \cdot, \cdot \rangle_\eta := \langle \cdot, \eta \cdot \rangle$, wrt which H Hermitian.

We find η for $q = \exp(i\pi/r)$ when

1. r integer > 2
2. $r \in R$ irrational $> N$.

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Algebraic structures

The Temperley-Lieb Algeba TL_N

Generated by $\{e_1, e_2, \cdots, e_{N-1}\}$ with relations

$$e_i^2 = -(q+q^{-1})e_i, \quad e_i e_{i\pm 1} e_i = e_i, \quad e_i e_j = e_j e_i \text{ for } |i-j|>1$$

The E_i form a representation of the Temperley-Lieb algebra.

The Hecke Algeba H_N

H_N algebra gen. by $\{b_1, b_2, \cdots, b_{N-1}\}$ with relations

$$(b_i+q)(b_i-q^{-1})=0, \quad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad b_i b_j = b_j b_i \quad |i-j|>1$$

There is a homomorphism $H_N \to TL_N, \quad b_i \mapsto e_i + q^{-1}$

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The quantum group $U_q(sl_2)$

Generated by $\{s^{\pm}, q^{\pm s^z}\}$ with relns

$$[s^+, s^-] = \frac{q^{2s^z} - q^{-2s^z}}{q - q^{-1}}, \quad q^{s^z} s^{\pm} q^{-s^z} = q^{\pm 1} s^{\pm}$$
$$q^{s^z} q^{-s^z} = 1 = q^{-s^z} q^{s^z}$$

There is a representation

$$S^z = \frac{1}{2}\sum_i \sigma_i, \; S^{\pm} = \sum_i q^{\frac{\sigma}{2}} \otimes \cdots \otimes q^{\frac{\sigma^z}{2}} \otimes \sigma_i^{\pm} \otimes q^{-\frac{\sigma^z}{2}} \otimes \cdots \otimes q^{-\frac{\sigma^z}{2}}$$

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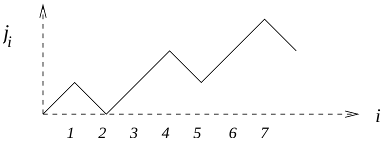
q generic

$V^{\otimes N}$ decomposes wrt $U_q(sl_2)$. e.g. for $N = 2$ we get a 'triplet' and 'singlet'.

Each irred repn $\rho_{\mathbf{j}}$ in $V^{\otimes N}$ is characterised by a path

$$\mathbf{j} = (j_1 = \frac{1}{2}, j_2, \cdots, j_N), \quad j_i \geq 0, \;\; j_{i+1} = j_i \pm 1.$$



Basis vectors

$|\mathbf{j}; m\rangle, \quad m \in \{j_N, j_N - 1, \cdots, -j_N\}$ all known explicitly

e.g. $N = 2$:

triplet $|\frac{1}{2}, 1; 1\rangle, \quad |\frac{1}{2}, 1; 0\rangle, \quad |\frac{1}{2}, 1; -1\rangle$ and singlet $|\frac{1}{2}, 0; 0\rangle$

Find $S^{\pm}|\mathbf{j}; m\rangle = c_{m,j_N}|\mathbf{j}; m \pm 1\rangle, \quad S^z|\mathbf{j}; m\rangle = q^m|\mathbf{j}; m\rangle$

and $E_i|\mathbf{j}; m\rangle = \delta_{j_{i-1}, j_{i+1}} \sum_{j'} d_{j_i, j'} |j_1, \cdots, \underset{i}{j'}, \cdots, j_N; m\rangle$

So $U_q(sl_2)$ and TL_N commute.

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So $U_q(sl_2)$ and TL_N commute.

$q^r = \pm 1$

When $q = \exp(i\pi/r)$, r integer, we find $(S^+)^r = 0$.

Consequences:

1. some $\rho_{\mathbf{j}}$ mix
2. some $\rho_{\mathbf{j}}$ reducible under action of $U_q(sl_2)$.
3. $H = \sum_{i=1}^{N} E_i$ not diagonalisable.

To fix all such problems for $r \geq 3$:
throw out all paths $\mathbf{j}$ with any $j_i \geq (r-1)/2$.

Called quantum group reduction.

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Conventional Approach to η

Find bi-orthormal set $|\phi_i\rangle$ and $|\psi_i\rangle$ s.t.

$$H|\phi_i\rangle = \lambda_i|\phi_i\rangle, \quad H^*|\psi_i\rangle = \lambda_i|\psi_i\rangle, \quad \langle\psi_i|\phi_j\rangle = \delta_{i,j}.$$

Then

$\eta = \sum_i |\psi_i\rangle\langle\psi_i|$ Hermitian, invertible, +ve and $\eta H = H^*\eta$

Works, but impractical.

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Algebraic approach to η

Choose bi-orthornormal set $|\mathbf{j};m\rangle$, $|\mathbf{j};m\rangle^T := T|\mathbf{j};m\rangle$ where $\mathbf{j}$ is restricted path.

By construction, satisfy

$$^T\langle\mathbf{j};m|\mathbf{j}';m'\rangle = \delta_{\mathbf{j},\mathbf{j}'}\delta_{m,m'}$$

Define: $\eta = \sum_{\mathbf{j},m} {}^T|\mathbf{j},m\rangle\langle\mathbf{j},m|^T$

We have $\eta|\mathbf{j},m\rangle = |\mathbf{j},m\rangle^T$ but linear. Properties:

1) $\eta^{-1} = \sum_{\mathbf{j},m} |\mathbf{j},m\rangle\langle\mathbf{j},m|$ 2) η Hermitian 3) η +ve.

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$\eta H = H^* \eta$ follows from:

Proposition: $\eta E_i |\mathbf{j}, m\rangle = E_i^* \eta |\mathbf{j}, m\rangle$

Proof: Recall

$E_i |\mathbf{j}; m\rangle = \delta_{j_{i-1}, j_{i+1}} \sum_{j'} d_{j_i, j'} |j_1, \cdots, j', \cdots, j_N; m\rangle.$

Crucial point is that $d_{j_i, j'}$ are real for restricted paths.

Hence

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C and C'

$\eta = \sum_{\mathbf{j},m} {}^T|\mathbf{j},m\rangle\langle\mathbf{j},m|^T$ explicit. More algebraic expression?

We have

- $\eta E_i = E_i^* \eta$
- $\eta S^{\pm} = (S^{\mp})^* \eta$

Hence $[\eta, TL_N] \neq 0$ and $[\eta, U_q(sl_2)] \neq 0$. Bummer.

However, simple calculation shows

- $RE_i = E_i^* R$ where $R(v_{s_1} \otimes \cdots \otimes v_{s_N}) = v_{-s_1} \otimes \cdots \otimes v_{-s_N}$
- $PS^{\pm} = (S^{\mp})^* P$

Hence 1) $[R\eta, TL_N] = 0$ which $\implies$ $[R\eta, H_N] = 0$

2) $[P\eta, U_q(sl_2)] = 0$.

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Defining $C' = R\eta$ and $C = P\eta$, identify

C' as element of $U_q(sl_2)$
C as element of H_N.

Further properties: $[C, C'] = 0$, $C^2 = C'^2 = 1$.

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$C'|\mathbf{j}; m\rangle = (-1)^{\frac{N}{2} - j_N}|\mathbf{j}; -m\rangle$

and we can write simply in terms of $S^{\pm}$ and S^z.

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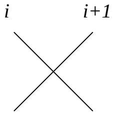
$\underline{C}$

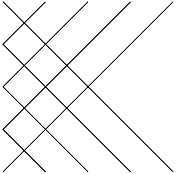
C satisfies $CE_i = E^*_{N-i}C$.

Also have $BE_i = E^*_{N-i}B$ where B is complete braid element of the Hecke Algebra
$B = \beta_N\beta_{N-1}\cdots\beta_1$ where $\beta_n = b_n b_{n-1}\cdots b_1$.

b_i



B



Thus CB^{-1} is central, and so constant on each TL_N irrep.

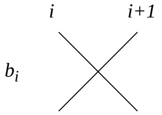
We find

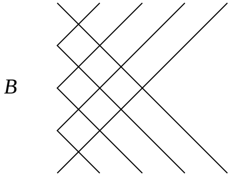
$$C = q^{\frac{N(N-4)}{4} - 2j_N(j_N+1)}B$$

<u>C</u>

C satisfies $CE_i = E^*_{N-i}C$.

Also have $BE_i = E^*_{N-i}B$ where B is complete braid element of the Hecke Algebra
$B = \beta_N \beta_{N-1} \cdots \beta_1$ where $\beta_n = b_n b_{n-1} \cdots b_1$.





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q not a root of unity

H believed to be diagonalisable with real eigenvalues $\forall q \in \mathbb{S}^1$.

Key requirement for our η construction was that $d_{j_i,j'}$ in

$$E_i|\mathbf{j}; m\rangle = \delta_{j_{i-1},j_{i+1}} \sum_{j'} d_{j_i,j'} |j_1, \cdots, j', \cdots, j_N; m\rangle$$

are real. We find

Real for $q^{i\pi/r}$, $r > N$ ✓

Not real for $q^{i\pi/r}$, $r < N$ $\times$

Summary and questions

- Explicit construction for η that doesn't require eigenvectors
- Natural to introduce $C = P\eta$ and $C' = R\eta$.
- C and C' have nice purely algebraic expressions.
- Other models, e.g. Pott's, have Hamiltonian expressed in terms of TL_N generators in similar way. Similar approach should be possible.
- What about q not a root of unity outside our wedge?

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