$\mathcal{PT}$-symmetry in nonlinear systems

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Wolfram Physics Project

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Outline

(1) $\mathcal{PT}$-symmetric quantum mechanics

(2) Nonlinear integrable systems

Why is Hermiticity a good property to have?

- Hermiticity ensures the reality of the energies

  Schrödinger equation $H|\psi\rangle = E|\psi\rangle$, $\langle\psi|H^\dagger = E^*\langle\psi|$

$$\left.\begin{array}{r} \langle\psi|\, H\,|\psi\rangle = E\,\langle\psi|\,\psi\rangle \\ \langle\psi|\, H^\dagger\,|\psi\rangle = E^*\,\langle\psi|\,\psi\rangle \end{array}\right\} \Rightarrow 0 = (E - E^*)\,\langle\psi|\,\psi\rangle$$

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- Hermiticity ensures conservation of probability densities

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$$

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- Thus when $H \neq H^\dagger$ one usually thinks of dissipation.
- However, these systems are in general open and do not possess a self-consistent description. (As much as QM is self-consistent.)

Both properties can be achieved in a non-Hermitian theory

- Wigner: Operators $\mathcal{O}$ which are left invariant under an antilinear involution $\mathcal{I}$ and whose eigenfunctions Φ also respect this symmetry,

$$[\mathcal{O}, \mathcal{I}] = 0 \quad \wedge \quad \mathcal{I}\Phi = \Phi$$

have a real eigenvalue spectrum.[a]

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- By defining a new metric also a consistent quantum mechanical framework has been developed for theories involving such operators.[b]

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[b] F. Scholtz, H. Geyer, F. Hahne, *Ann. Phys.* 213 (1992) 74
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In particular this also holds for $\mathcal{O}$ being non-Hermitian.

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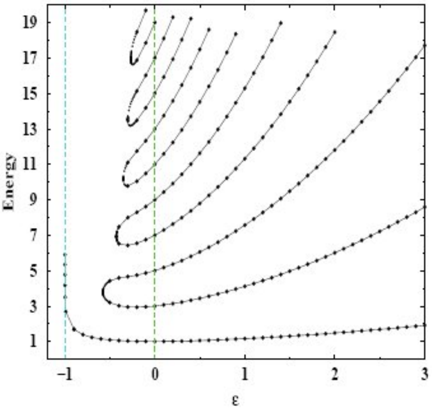
The seminal classical example

$$\mathcal{H} = \frac{1}{2}p^2 + x^2(ix)^{\varepsilon} \qquad \text{for } \varepsilon \in \mathbb{R}$$



- real eigenvalues for $\varepsilon \geq 0$
- exceptional points for $\varepsilon < 0$

C.M. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243

Lattice Reggeon field theory

$$\mathcal{H} = \sum_{\vec{\imath}} \left[\Delta a^{\dagger}_{\vec{\imath}} a_{\vec{\imath}} + i g a^{\dagger}_{\vec{\imath}} (a_{\vec{\imath}} + a^{\dagger}_{\vec{\imath}}) a_{\vec{\imath}} + \tilde{g} \sum_{\vec{\jmath}} (a^{\dagger}_{\vec{\imath}+\vec{\jmath}} - a^{\dagger}_{\vec{\imath}})(a_{\vec{\imath}+\vec{\jmath}} - a_{\vec{\imath}}) \right]$$

- $a^{\dagger}_{\vec{\imath}}, a_{\vec{\imath}}$ are creation and annihilation operators, $\Delta, g, \tilde{g} \in \mathbb{R}$ [a]

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- $a^\dagger_{\vec{\imath}}, a_{\vec{\imath}}$ are creation and annihilation operators, $\Delta, g, \tilde{g} \in \mathbb{R}$ [a]
- for one site this is almost ix^3

$$\begin{aligned} \mathcal{H} &= \Delta a^\dagger a + iga^\dagger \left(a + a^\dagger \right) a \\ &= \frac{1}{2} \left(\hat{p}^2 + \hat{x}^2 - 1 \right) + i \frac{g}{\sqrt{2}} (\hat{x}^3 + \hat{p}^2 \hat{x} - 2\hat{x} + i\hat{p}) \end{aligned}$$

with $a = (\omega \hat{x} + i\hat{p})/\sqrt{2\omega}$, $a^\dagger = (\omega \hat{x} - i\hat{p})/\sqrt{2\omega}$ [b]

[a] J.L. Cardy, R. Sugar, *Phys. Rev.* D12 (1975) 2514
[b] P. Assis, A. Fring, J. Phys. A41 (2008) 244001

Quantum spin chains (c=-22/5 CFT)

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} \sigma_i^x + \lambda\sigma_i^z\sigma_{i+1}^z + ih\sigma_i^z \quad \lambda, h \in \mathbb{R}$$

G. von Gehlen, J. Phys. A24 (1991) 5371

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Field theories

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta^2}\sum_{k=\mathbf{a}}^{\ell} n_k \exp(\beta\alpha_k\cdot\phi)$$

$a = 1 \equiv$ conformal Toda field theory (Lie algebras)

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Strings on $AdS_5 \times S^5$-background

A. Das, A. Melikyan, V. Rivelles, *JHEP* 09 (2007) 104

Deformed space-time structures

- deformed Heisenberg canonical commutation relations

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$$X = \alpha a^\dagger + \beta a, \quad P = i\gamma a^\dagger - i\delta a, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$\begin{aligned}[X, P] &= i\hbar q^{g(N)}(\alpha\delta + \beta\gamma) \\ &\quad + \frac{i\hbar(q^2-1)}{\alpha\delta+\beta\gamma}\left(\delta\gamma X^2 + \alpha\beta\, P^2 + i\alpha\delta XP - i\beta\gamma PX\right)\end{aligned}$$

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- limit: $\beta \to \alpha$, $\delta \to \gamma$, $g(N) \to 0$, $q \to e^{2\tau\gamma^2}$, $\gamma \to 0$

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- representation: $X = (1 + \tau p_0^2)x_0$, $P = p_0$, $[x_0, p_0] = i\hbar$

- with the standard inner product X is not Hermitian

$$X^\dagger = X + 2\tau i\hbar P \quad \text{and} \quad P^\dagger = P$$

B. Bagchi and A. Fring, Phys. Lett. A373 (2009) 4307

D. Dey, A. Fring, B. Khantoul, J. Phys. A: Math. and Theor. 46 (2013) 335304

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$$X^\dagger = X + 2\tau i\hbar P \quad and \quad P^\dagger = P$$

- $\Rightarrow H(X, P)$ is in general not Hermitian

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- example harmonic oscillator:

$$\begin{aligned} H_{ho} &= \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}(1+\tau p_0^2)x_0(1+\tau p_0^2)x_0 \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}\left[(1+\tau p_0^2)^2 x_0^2 + 2i\hbar\tau p_0(1+\tau p_0^2)x_0\right] \end{aligned}$$

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- but also Hermitian representations exist:

$$X = x_0 \quad and \quad P = \frac{1}{\sqrt{\tau}}\tan(\sqrt{\tau}p_0)$$

B. Bagchi and A. Fring, Phys. Lett. A373 (2009) 4307
D. Dey, A. Fring, B. Khantoul, J. Phys. A: Math. and Theor. 46 (2013) 335304

Dynamical noncommutative space-time

$$
\begin{array}{lll}
[x_0, y_0] = i\theta, & [x_0, p_{x_0}] = i\hbar, & [y_0, p_{y_0}] = i\hbar, \\
[p_{x_0}, p_{y_0}] = 0, & [x_0, p_{y_0}] = 0, & [y_0, p_{x_0}] = 0,
\end{array}
$$

replaced by ($\theta \in \mathbb{R}$)

$$
\begin{array}{ll}
[X, Y] = i\theta(1 + \tau Y^2) & [X, P_x] = i\hbar(1 + \tau Y^2) \\
[Y, P_y] = i\hbar(1 + \tau Y^2) & [X, P_y] = 2i\tau Y(\theta P_y + \hbar X) \\
[P_x, P_y] = 0 & [Y, P_x] = 0
\end{array}
$$

$\Rightarrow$ Non-Hermitian representation

$$
X = (1 + \tau y_0^2)x_0 \quad Y = y_0 \quad P_x = p_{x_0} \quad P_y = (1 + \tau y_0^2)p_{y_0}
$$

$$
X^\dagger = X + 2i\tau\theta Y \quad Y^\dagger = Y \quad P_y^\dagger = P_y - 2i\tau\hbar Y \quad P_x^\dagger = P_x
$$

A. Fring, L. Gouba, F. Scholtz, J. Phys. A: Math and Theor. 43 (2010) 345401
A. Fring, L. Gouba, B. Bagchi, J. Phys. A: Math and Theor. 43 (2010) 425202

How to explain the reality of the spectrum?

1. Pseudo/Quasi-Hermiticity
2. $\mathcal{PT}$-symmetry
3. Supersymmetry (Darboux transformations)

Pseudo/Quasi-Hermiticity

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \qquad \rho = \eta^\dagger \eta \qquad (*)$$

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$$h\phi = E\phi \Rightarrow \eta H \eta^{-1}\phi = E\phi \Rightarrow H\eta^{-1}\phi = E\eta^{-1}\phi \Rightarrow H\psi = E\psi \;\; \psi := \eta^{-1}\phi$$

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	$H^{\dagger} = \rho H \rho^{-1}$	$H^{\dagger}\rho = \rho H$	$H^{\dagger} = \rho H \rho^{-1}$
positivity of ρ	$\checkmark$	$\checkmark$	$\times$
ρ Hermitian	$\checkmark$	$\checkmark$	$\checkmark$
ρ invertible	$\checkmark$	$\times$	$\checkmark$
terminology	(*)	quasi-Herm. [a]	pseudo-Herm.[b]
spectrum of H	real	could be real	real
definite metric	guaranteed	guaranteed	not conclusive

[a] J. Dieudonné, Proc. Int. Symp. (1961) 115
F. Scholtz, H. Geyer, F. Hahne, Ann. Phys. 213 (1992) 74

[b] M. Froissart, Nuovo Cim. 14 (1959) 197
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Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
  $(\mathcal{P}: x \to -x, p \to -p;\ \mathcal{T}: x \to x, p \to -p, i \to -i)$

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- $\mathcal{PT}$ is an anti-linear operator:

$$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi \qquad \lambda, \mu \in \mathbb{C}$$

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$$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \quad \Rightarrow \varepsilon = \varepsilon^* \ \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$

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- *Proof*:

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- Real eigenvalues from unbroken $\mathcal{PT}$-symmetry:

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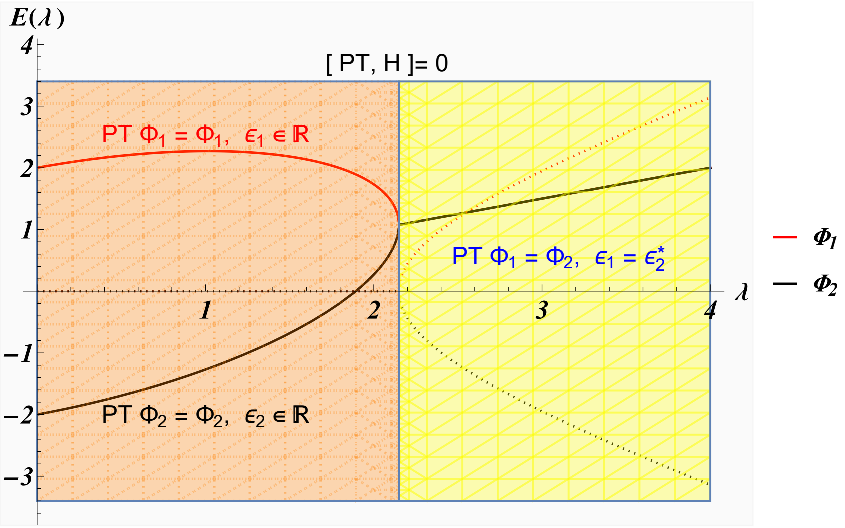
$\mathcal{PT}$-symmetry is only an example of an antilinear operator.

$\mathcal{PT}$-symmetry versus spontaneously broken $\mathcal{PT}$-symmetry



real parts are solid lines, imaginary parts are dotted lines

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Decompose Hamiltonian $\mathcal{H}$ as:

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  $\Rightarrow$isospectral Hamiltonians

$$H_\pm^m = -\Delta + V_\pm^m + E_m \quad H_\pm^m \Phi_n^\pm = E_n \Phi_n^\pm \quad \text{for } n > m$$

H_-^m non-Hermitian and H_+^m Hermitian when $ReW = \frac{1}{2}\partial_x \ln(ImW)$.

How to formulate a quantum mechanical framework?

1. orthogonality
2. observables
3. uniqueness
4. technicalities (new metric etc)

Orthogonality

- Take h to be a Hermitian and diagonalisable Hamiltonian:

$$\langle \phi_n | h\phi_m \rangle = \langle h\phi_n | \phi_m \rangle$$

$$\begin{aligned} |h\phi_m\rangle &= \varepsilon_m \quad |\phi_m\rangle \\ \langle h\phi_n| &= \varepsilon_n^* \langle \phi_n| \end{aligned}$$

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- Take H to be a non-Hermitian Hamiltonian:

$$H |\Phi_n\rangle = \varepsilon_n |\Phi_n\rangle$$

- reality and orthogonality no longer guaranteed. Define

$$\langle \Phi_n | \Phi_m \rangle_\eta := \langle \Phi_n | \eta^2 \Phi_m \rangle$$

- where $\quad \langle \Phi_n | H\Phi_m \rangle_\eta = \langle H\Phi_n | \Phi_m \rangle_\eta \Rightarrow \langle \Phi_n | \Phi_m \rangle_\eta = \delta_{n,m}$

H is Hermitian with respect to new metric

- Assume pseudo-Hermiticity:

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Using the same reasoning as in the Hermitian case:
$\Rightarrow$ Eigenvalues of *H* are real, eigenstates are orthogonal

Observables

- Observables are associated to self-adjoint (Hermitian) operators

$$\langle \psi | o\phi \rangle = \langle o\psi | \phi \rangle$$

- Observables in the non-Hermitian system are associated to self-adjoint (Hermitian) operators $\mathcal{O}$ with a re-defined metric

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Examples: In $\mathcal{H} = \frac{1}{2}p^2 + ix^3$ x, p are not observables,
but $X = \eta^{-1} x \eta$, $P = \eta^{-1} p \eta$ are.

General technique, construction of metric and Dyson maps

- Given $H \begin{cases} \text{either} & \text{solve } \eta H \eta^{-1} = h \quad \text{for} \quad \eta \Rightarrow \rho = \eta^\dagger \eta \\ \text{or} & \text{solve } H^\dagger = \rho H \rho^{-1} \quad \text{for} \quad \rho \Rightarrow \eta = \sqrt{\rho} \end{cases}$

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- Ambiguities:

  Given H the metric is not uniquely defined for unknown h.

  $\Rightarrow$ Given only H the observables are not uniquely defined.
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  - Fixing one more observable achieves uniqueness. [a]

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Note:

- Thus, this is not re-inventing or disputing the validity of quantum mechanics. We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

An example with a finite dimensional Hilbert space:

Two-level system

$$H = -\frac{1}{2}\left[\omega\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right]$$

with eigensystem

$$E_\pm = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \qquad \varphi_\pm = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}$$

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An example with a finite dimensional Hilbert space:

Two-level system

$$H = -\frac{1}{2}\left[\omega \mathbb{I} + \lambda \sigma_z + i\kappa\sigma_x\right]$$

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$$[\mathcal{PT}, H] = 0, \quad \text{and} \quad \mathcal{PT}\varphi_{\pm} = -\varphi_{\pm} \qquad \text{for} \quad |\lambda| > |\kappa|$$

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Claim: This system has real energies for $|\lambda(t)| < |\kappa(t)|$!

$\mathcal{PT}$ symmetrically coupled harmonic oscillator (∞- dim Hilbert space)

$$H_K = aK_1 + bK_2 + i\lambda K_3, \qquad a, b, \lambda \in \mathbb{R}$$

with Lie algebraic generators

$$\begin{aligned} K_1 &= \left(p_x^2 + x^2\right)/2, \;\; K_2 = \left(p_y^2 + y^2\right)/2, \;\; K_3 = \left(xy + p_x p_y\right)/2 \\ K_4 &= \left(xp_y - yp_x\right)/2 \end{aligned}$$

$$\begin{array}{lll} [K_1, K_2] = 0, & [K_1, K_3] = iK_4, & [K_1, K_4] = -iK_3, \\ [K_2, K_3] = -iK_4, & [K_2, K_4] = iK_3, & [K_3, K_4] = i(K_1 - K_2)/2 \end{array}$$

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- H_K is quasi-Hermitian: $h_K = \eta H_K \eta^{-1}$

$$h_K = (a+b)\left(K_1 + K_2\right)/2 + \sqrt{(a-b)^2 - \lambda^2}\left(K_1 - K_2\right)/2$$

Dyson map: $\eta = e^{2\theta K_4}$, $\theta = \operatorname{arctanh}[\lambda/(b-a)]$, $\mathcal{PT}$-symm. $|\lambda| < |a-b|$

Theoretical framework (key equations)

Time-dependent Schrödinger eqn for $h(t) = h^{\dagger}(t)$, $H(t) \neq H^{\dagger}(t)$

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \qquad \text{and} \qquad H(t)\Psi(t) = i\hbar\partial_t\Psi(t)$$

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$$\phi(t) = \eta(t)\Psi(t)$$

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$\Rightarrow$ Time-dependent quasi-Hermiticity relation

$$H^{\dagger}\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t)$$

[from conjugating Dyson relation and $\rho(t) := \eta^{\dagger}(t)\eta(t)$)]

The Hamiltonian $H(t)$ is nonobservable and not the energy operator

Recall: Observables $o(t)$ in the Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi Hermitian

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A. Fring, T. Frith, *Phys. Rev. A* 95 (2017) 010102

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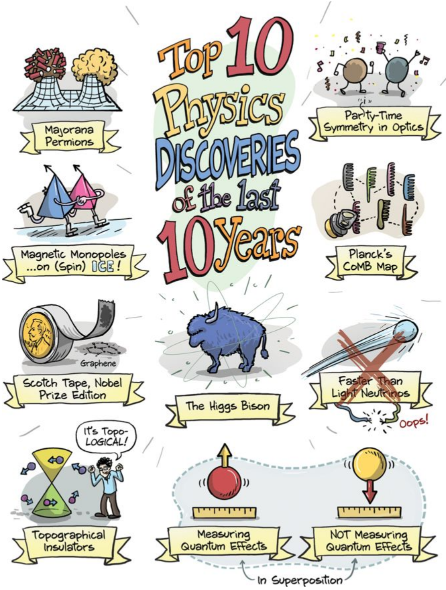
Since $H(t)$ is not quasi/pseudo Hermitian it is not an observable.
The observable energy operator is

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t).$$

A. Fring, T. Frith, *Phys. Rev. A* 95 (2017) 010102

Nature Physics volume 11, page 799 (2015)



Helmholtz equation
in paraxial approximation:

$$i\frac{\partial\psi}{\partial z}+\frac{1}{2k}\frac{\partial^2\psi}{\partial x^2}+kv(x)\psi=0$$

$\psi \equiv$ envelope function of E
$v(x) = n/n_0 - 1$
$n \equiv$ reflection index
$n_0 \equiv$ reflection index
$k = n\omega/c$
$\omega \equiv$ frequency

with $z \to t$
this becomes formally
the Schrödinger equation

Time-dependent coupled oscillators

$$H(t) = \frac{a(t)}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + i\frac{\lambda(t)}{2}\left(xy + p_x p_y\right), \; a(t), \lambda(t) \in \mathbb{R}$$

Ansatz:

$$\eta(t) = \prod\nolimits_{i=1}^{4} e^{\gamma_i(t) K_i}, \qquad \gamma_i \in \mathbb{R}$$

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Time-dependent Dyson equations is satisfied when

Constraint:

$$\gamma_1 = \gamma_2 = q_1, \quad \dot{\gamma}_3 = -\lambda \cosh \gamma_4, \quad \dot{\gamma}_4 = \lambda \tanh \gamma_3 \sinh \gamma_4,$$

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Solution: $\gamma_4 = \operatorname{arcsinh}\left(\kappa \operatorname{sech} \gamma_3\right), \; \chi(t) := \cosh \gamma_3, \; \kappa = const$

with dissipative Ermakov-Pinney equation

$$\ddot{\chi} - \frac{\dot{\lambda}}{\lambda}\dot{\chi} - \lambda^2 \chi = \frac{\kappa^2 \lambda^2}{\chi^3}$$

Instanteneous energies are real even in the broken $\mathcal{PT}$ regime !

Von Neumann entropy in $\mathcal{PT}$-symmetric systems

statistical ensemble of states (density matrix):

$$\varrho_h = \sum\nolimits_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|$$

partial traces (for subsystems)

$$\begin{aligned} \varrho_{h,A} &= \mathrm{Tr}_B(\varrho_h) = \sum\nolimits_i \left\langle n_{i,B}\right| \varrho_h \left|n_{i,B}\right\rangle \\ \varrho_{h,B} &= \mathrm{Tr}_A(\varrho_h) = \sum\nolimits_i \left\langle n_{i,A}\right| \varrho_h \left|n_{i,A}\right\rangle \end{aligned}$$

$\left|n_{i,A}\right\rangle, \left|n_{i,B}\right\rangle \equiv$ eigenstates of the subsystems A, B

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Time evolution:

$$i\partial_t \varrho_h = [h, \varrho_h]$$

It follows

$$i\partial_t \varrho_H = [h, \varrho_H]$$

with

$$\varrho_h = \eta \varrho_H \eta^{-1}, \quad h = \eta H \eta^{-1} + i\partial_t \eta \eta^{-1}$$

Therefore

$$\varrho_H = \sum\nolimits_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \rho$$

recalling that $\rho = \eta^\dagger \eta$, $\left|\phi_i\right\rangle = \eta \left|\psi_i\right\rangle$

Von Neumann entropy

$$S_h = -\operatorname{tr}\left[\rho_h \ln \rho_h\right] = -\sum\nolimits_i \lambda_i \ln \lambda_i = S_H$$

Entropy of a subsystem

$$S_{h,A} = -\operatorname{tr}\left[\varrho_{h,A} \ln \varrho_{h,A}\right] = -\sum\nolimits_i \lambda_{i,A} \ln \lambda_{i,A} = S_{H,A}$$

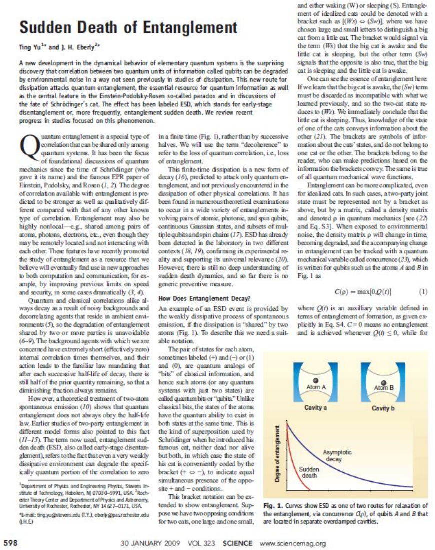
An example: bosonic system coupled to a bath

$$H = \nu a^\dagger a + \nu \sum_{n=1}^{N} q_n^\dagger q_n + (g+\kappa) a^\dagger \sum_{n=1}^{N} q_n + (g-\kappa) a \sum_{n=1}^{N} q_n^\dagger$$

energy eigenvalues

$$E_{m,N}^{\pm} = m\left(\nu \pm \sqrt{N}\sqrt{g^2-\kappa^2}\right)$$

Standard behaviour:



Sudden Death of Entanglement

Ting Yu[1*] and J. H. Eberly[2*]

A new development in the dynamical behavior of elementary quantum systems is the surprising discovery that correlation between two quantum units of information called qubits can be degraded by environmental noise in a way not seen previously in studies of dissipation. This new route for dissipation attacks quantum entanglement, the essential resource for quantum information as well as the central feature in the Einstein-Podolsky-Rosen so-called paradox and in discussions of the fate of Schrödinger's cat. The effect has been labeled ESD, which stands for early-stage disentanglement or, more frequently, entanglement sudden death. We review recent progress in studies focused on this phenomenon.

Quantum entanglement is a special type of correlation that can be shared only among quantum systems. It has been the focus of foundational discussions of quantum mechanics since the time of Schrödinger (who gave it its name) and the famous EPR paper of Einstein, Podolsky, and Rosen (*1*, *2*). The degree of correlation available with entanglement is predicted to be stronger as well as qualitatively different compared with that of any other known type of correlation. Entanglement may also be highly nonlocal—e.g., shared among pairs of atoms, photons, electrons, etc., even though they may be remotely located and not interacting with each other. These features have recently promoted the study of entanglement as a resource that we believe will eventually find use in new approaches to both computation and communication, for example, by improving previous limits on speed and security, in some cases dramatically (*3*, *4*).

Quantum and classical correlations alike always decay as a result of noisy backgrounds and decorrelating agents that reside in ambient environments (*5*), so the degradation of entanglement shared by two or more parties is unavoidable (*6*–*9*). The background agents with which we are concerned have extremely short (effectively zero) internal correlation times themselves, and their action leads to the familiar law mandating that after each successive half-life of decay, there is still half of the prior quantity remaining, so that a diminishing fraction always remains.

However, a theoretical treatment of two-atom spontaneous emission (*10*) shows that quantum entanglement does not always obey the half-life law. Earlier studies of two-party entanglement in different model forms also pointed to this fact (*11*–*15*). The term now used, entanglement sudden death (ESD, also called early-stage disentanglement), refers to the fact that even a very weakly dissipative environment can degrade the specifically quantum portion of the correlation to zero in a finite time (Fig. 1), rather than by successive halves. We will use the term "decoherence" to refer to the loss of quantum correlation, i.e., loss of entanglement.

This finite-time dissipation is a new form of decay (*16*), predicted to attack only quantum entanglement, and not previously encountered in the dissipation of other physical correlations. It has been found in numerous theoretical examinations to occur in a wide variety of entanglements involving pairs of atomic, photonic, and spin qubits, continuous Gaussian states, and subsets of multiple qubits and spin chains (*17*). ESD has already been detected in the laboratory in two different contexts (*18*, *19*), confirming its experimental reality and supporting its universal relevance (*20*). However, there is still no deep understanding of sudden death dynamics, and so far there is no generic preventive measure.

How Does Entanglement Decay?

An example of an ESD event is provided by the weakly dissipative process of spontaneous emission, if the dissipation is "shared" by two atoms (Fig. 1). To describe this we need a suitable notation.

The pair of states for each atom, sometimes labeled (+) and (−) or (1) and (0), are quantum analogs of "bits" of classical information, and hence such atoms (or any quantum systems with just two states) are called quantum bits or "qubits." Unlike classical bits, the states of the atoms have the quantum ability to exist in both states at the same time. This is the kind of superposition used by Schrödinger when he introduced his famous cat, neither dead nor alive but both, in which case the state of his cat is conveniently coded by the bracket (+ ⇔ −), to indicate equal simultaneous presence of the opposite + and − conditions.

This bracket notation can be extended to show entanglement. Suppose we have two opposing conditions for two cats, one large and one small, and either waking (W) or sleeping (S). Entanglement of idealized cats could be denoted with a bracket such as [(*Ws*) ⇔ (*Sw*)], where we have chosen large and small letters to distinguish a big cat from a little cat. The bracket would signal via the term (*Ws*) that the big cat is awake and the little cat is sleeping, but the other term (*Sw*) signals that the opposite is also true, that the big cat is sleeping and the little cat is awake.

One can see the essence of entanglement here: If we learn that the big cat is awake, the (*Sw*) term must be discarded as incompatible with what we learned previously, and so the two-cat state reduces to (*Ws*). We immediately conclude that the little cat is sleeping. Thus, knowledge of the state of one of the cats conveys information about the other (*21*). The brackets are symbols of information about the cats' states, and do not belong to one cat or the other. The brackets belong to the reader, who can make predictions based on the information the brackets convey. The same is true of all quantum mechanical wave functions.

Entanglement can be more complicated, even for idealized cats. In such cases, a two-party joint state must be represented not by a bracket as above, but by a matrix, called a density matrix and denoted ρ in quantum mechanics [see (*22*) and Eq. S3]. When exposed to environmental noise, the density matrix ρ will change in time, becoming degraded, and the accompanying change in entanglement can be tracked with a quantum mechanical variable called concurrence (*23*), which is written for qubits such as the atoms A and B in Fig. 1 as

$$C(\rho) = \max[0, Q(t)] \qquad (1)$$

where $Q(t)$ is an auxiliary variable defined in terms of entanglement of formation, as given explicitly in Eq. S4. $C = 0$ means no entanglement and is achieved whenever $Q(t) \leq 0$, while for

Fig. 1. Curves show ESD as one of two routes for relaxation of the entanglement, via concurrence $C(\rho)$, of qubits A and B that are located in separate overdamped cavities.

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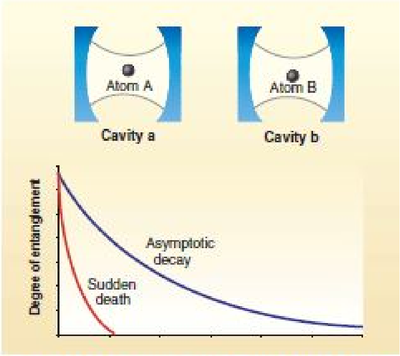
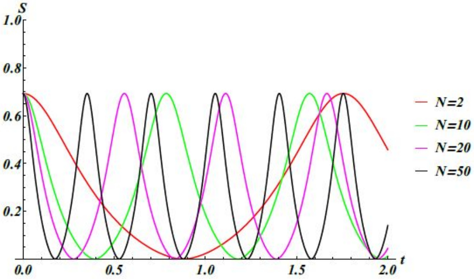
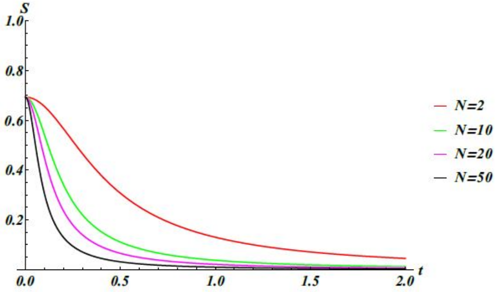


Fig. 1. Curves show ESD as one of two routes for relaxation of

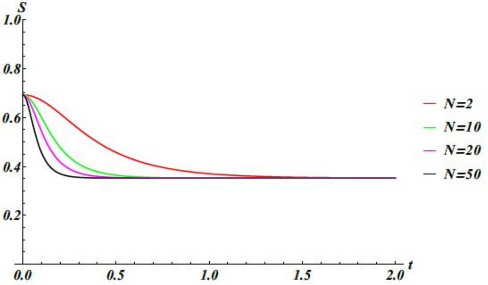
Von-Neumann entropy in the $\mathcal{PT}$ symmetric regime



Von-Neumann entropy at the exceptional point



Von-Neumann entropy in the broken $\mathcal{PT}$ regime



For more detail on this part of the talk see
A.Fring,"An introduction to PT-symmetric quantum mechanics – time-dependent systems." arXiv:2201.05140 (2022).

Reality of N-Soliton charges

The complex KdV equation equals two coupled real equations

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Leftrightarrow \quad \begin{cases} p_t + 6pp_x + p_{xxx} - 6qq_x = 0 \\ q_t + 6\left(pq\right)_x + q_{xxx} = 0 \end{cases}$$

with $u(x,t) = p(x,t) + iq(x,t)$, $p(x,t)$, $q(x,t) \in \mathbb{R}$

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- Unifies some know special cases:
  - for $(pq)_x \to pq_x$: complex KdV $\Rightarrow$ Hirota-Satsuma equations
  - for $q_{xxx} \to 0$ complex KdV $\Rightarrow$ Ito equations

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- Integrability:
  Lax pair:

$$L_t = [M, L] \quad L = \partial_x^2 + \frac{1}{6}u, \;\; M = 4\partial_x^3 + u\partial_x + \frac{1}{2}u_x$$

Solutions from Hirota's direct method

Convert KdV equation into Hirota's bilinear form

$$\left(D_x^4 + D_x D_t\right) \tau \cdot \tau = 0$$

with $u = 2(\ln \tau)_{xx}$. (D_x, D_t are Hirota derivatives)

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with $u = 2(\ln\tau)_{xx}$. (D_x, D_t are Hirota derivatives)
Expanding $\tau = \sum_{k=0}^{\infty}\lambda^k\tau^k$ gives multi-soliton solutions

$$\begin{aligned}
\tau_{\mu;\alpha}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} \\
\tau_{\mu,\nu;\alpha,\beta}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + \varkappa(\alpha,\beta)e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
\tau_{\mu,\nu,\rho;\alpha,\beta,\gamma}(x,t) &= 1 + e^{\eta_{\mu;\alpha}} + e^{\eta_{\nu;\beta}} + e^{\eta_{\rho;\gamma}} + \varkappa(\alpha,\beta)e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}} \\
&\quad +\varkappa(\alpha,\gamma)e^{\eta_{\mu;\alpha}+\eta_{\rho;\gamma}} + \varkappa(\beta,\gamma)e^{\eta_{\nu;\beta}+\eta_{\rho;\gamma}} \\
&\quad +\varkappa(\alpha,\beta)\varkappa(\alpha,\gamma)\varkappa(\beta,\gamma)e^{\eta_{\mu;\alpha}+\eta_{\nu;\beta}+\eta_{\rho;\gamma}}
\end{aligned}$$

with $\eta_{\mu;\alpha} := \alpha x - \alpha^3 t + \mu$, $\varkappa(\alpha,\beta) := (\alpha-\beta)^2/(\alpha+\beta)^2$
$\mu,\nu,\rho \in \mathbb{C}$, $\alpha,\beta,\gamma \in \mathbb{R}$

One-soliton solution

We find

$$u_{i\theta;\alpha}(x,t) = \frac{\alpha^2 + \alpha^2 \cos\theta \cosh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2} - i \frac{\alpha^2 \sin\theta \sinh(\alpha x - \alpha^3 t)}{[\cos\theta + \cosh(\alpha x - \alpha^3 t)]^2}$$

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The solution found by Khare and Saxena is the special case

$$u_{\pm i\frac{\pi}{2};\alpha}(x,t) = \alpha^2 \mathrm{sech}^2\left(\alpha x - \alpha^3 t\right) \mp i\alpha^2 \tanh\left(\alpha x - \alpha^3 t\right) \mathrm{sech}\left(\alpha x - \alpha^3 t\right)$$

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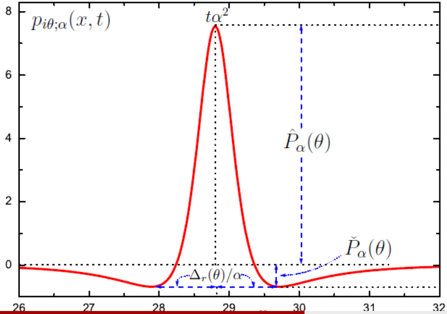

$$\begin{aligned} \hat{P}_\alpha(\theta) &= \frac{\alpha^2}{2}\sec^2\left(\frac{\theta}{2}\right) \\ \check{P}_\alpha(\theta) &= \frac{\alpha^2}{4}\cot^2(\theta) \\ \Delta_r(\theta) &= \text{arccosh}(\cos\theta - 2\sec\theta) \end{aligned}$$

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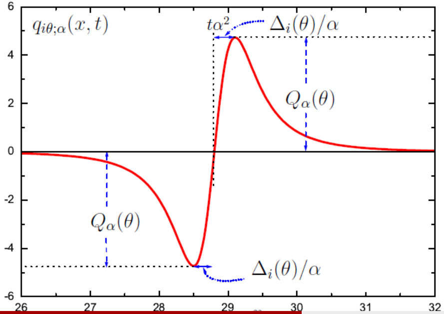

$$Q_\alpha(\theta) = \frac{8\alpha^2\sqrt{5+\cos(2\theta)+\cos\theta A}}{[6\cos\theta + A]^2 / \sin\theta}$$

$$\Delta_i(\theta) = \operatorname{arccosh}\left[\frac{1}{2}\cos\theta + \frac{1}{4}A\right]$$

$$A = \sqrt{2}\sqrt{17+\cos(2\theta)}$$

Real charges from one-soliton solution

$$
\begin{aligned}
\text{Mass} : m_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}(x,t)dx = 2\alpha \\
\text{Momentum} : p_\alpha &= \int_{-\infty}^{\infty} u_{i\theta;\alpha}^2 dx = \frac{2}{3}\alpha^3 \\
\text{Energy} : E_\alpha &= \int_{-\infty}^{\infty} \left[2u_{i\theta;\alpha}^3 - (u_{i\theta;\alpha})_x^2\right] dx = \frac{2}{5}\alpha^5
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Generic: $I_n = \int_{-\infty}^{\infty} w_{2n-2}(x,t)dx = \frac{2}{2n-1}\alpha^{2n-1}$

Reality follows immediately from $\mathcal{PT}$-symmetry

$E = \int_{-\infty}^{\infty} dx \mathcal{H}[\phi[x]] = -\int_{\infty}^{-\infty} dx \mathcal{H}[\phi[-x]] = \int_{-\infty}^{\infty} dx \mathcal{H}^\dagger[\phi[x]] = E^*$

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This is not possible for N-soliton solutions with $N > 2$.

Reality of complex N-soliton charges

Asymptotically complex N-solitons factor into N one-solitons

Charges based on one-solitons solutions are real by $\mathcal{PT}$-symmetry

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Therefore

Reality condition

$\mathcal{PT}$-symmetry and integrability ensure the reality of all charges.

Nonlocality

Consider higher order nonlinear Schrödinger equation

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$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\varepsilon \left[\alpha q_{xxx} + \beta |q|^2 q_x + \gamma q \, |q|_x^2 \right] = 0$$

$\mathcal{PT}$-symmetry: $\mathcal{PT} : x \to -x,\ t \to -t,\ i \to -i,\ q \to q$

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Integrable cases:

$\varepsilon = 0 \equiv$ nonlinear Schrödinger equation (NLSE)
$\alpha : \beta : \gamma = 0 : 1 : 1 \equiv$ derivative NLSE of type I
$\alpha : \beta : \gamma = 0 : 1 : 0 \equiv$ derivative NLSE of type II
$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation

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$\alpha : \beta : \gamma = 1 : 6 : 3 \equiv$ Sasa-Satsuma equation
$\alpha : \beta : \gamma = 1 : 6 : 0 \equiv$ Hirota equation

$$iq_t + \frac{1}{2} q_{xx} + |q|^2 q + i\varepsilon \left[q_{xxx} + 6 |q|^2 q_x \right] = 0$$

Zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0$$

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$$U = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & -A(x,t) \end{pmatrix}$$

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$$\begin{aligned} A_x(x,t) &= q(x,t)C(x,t) - r(x,t)B(x,t) \\ B_x(x,t) &= q_t(x,t) - 2q(x,t)A(x,t) - 2i\lambda B(x,t) \\ C_x(x,t) &= r_t(x,t) + 2r(x,t)A(x,t) + 2i\lambda C(x,t) \end{aligned}$$

$$\begin{aligned} A(x,t) &= -i\alpha qr - 2i\alpha\lambda^2 + \beta\left(rq_x - qr_x - 4i\lambda^3 - 2i\lambda qr\right) \\ B(x,t) &= i\alpha q_x + 2\alpha\lambda q + \beta\left(2q^2 r - q_{xx} + 2i\lambda q_x + 4\lambda^2 q\right) \\ C(x,t) &= -i\alpha r_x + 2\alpha\lambda r + \beta\left(2qr^2 - r_{xx} - 2i\lambda r_x + 4\lambda^2 r\right) \end{aligned}$$

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$$\begin{aligned} q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta\left[q_{xxx} - 6qrq_x\right] &= 0 \\ r_t + i\alpha r_{xx} - 2i\alpha qr^2 + \beta\left(r_{xxx} - 6qrr_x\right) &= 0 \end{aligned}$$

Nonlocality from zero curvature condition

Complex conjugate pair: $r(x,t) = \kappa q^*(x,t)$ (Hirota equation)

$$\begin{aligned} iq_t &= -\alpha\left(q_{xx} - 2\kappa\left|q\right|^2 q\right) - i\beta\left(q_{xxx} - 6\kappa\left|q\right|^2 q_x\right) \\ -iq_t^* &= -\alpha\left(q_{xx}^* - 2\kappa\left|q\right|^2 q^*\right) + i\beta\left(q_{xxx}^* - 6\kappa\left|q\right|^2 q_x^*\right) \end{aligned}$$

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$\mathcal{P}$ conjugate pair: $r(x,t) = \kappa q^*(-x,t)$ (Nonlocal Hirota equ^n)

$$\begin{aligned} iq_t &= -\alpha\left[q_{xx} - 2\kappa\tilde{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\tilde{q}^* q_x] \\ -i\tilde{q}_t^* &= -\alpha\left[\tilde{q}_{xx}^* - 2\kappa q(\tilde{q}^*)^2\right] - \delta(\tilde{q}_{xxx}^* - 6\kappa\tilde{q}^* q\tilde{q}_x^*) \end{aligned}$$

$$\beta = i\delta,\ \alpha,\delta \in \mathbb{R},\ q := q(x,t);\ \tilde{q} := q(-x,t)$$

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$\mathcal{T}$ conjugate pair: $r(x,t) = \kappa q^*(x,-t)$

$$\begin{aligned} iq_t &= -i\hat{\delta}\left[q_{xx} - 2\kappa \hat{q}^* q^2\right] + \delta[q_{xxx} - 6\kappa q\hat{q}^* q_x] \\ i\hat{q}_t^* &= i\hat{\delta}\left[\hat{q}_{xx}^* - 2\kappa q(\hat{q}^*)^2\right] + \delta(\hat{q}_{xxx}^* - 6\kappa\hat{q}^* q\hat{q}_x^*) \end{aligned}$$

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$$
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$\mathcal{P}$ transformed pair: $r(x,t) = \kappa q(-x,t)$:

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Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation ($q = g/f$)

$$f^3 \left[iq_t + \alpha q_{xx} - 2\kappa\alpha \left|q\right|^2 q + i\beta\Big(q_{xxx} - 6\kappa \left|q\right|^2 q_x\Big)\right] =$$

$$f\left[iD_t g\cdot f + \alpha D_x^2 g\cdot f + i\beta D_x^3 g\cdot f\right] + \left[3i\beta\left(\frac{g}{f}f_x - g_x\right) - \alpha g\right]$$

$$\times\left[D_x^2 f\cdot f + 2\kappa\left|g\right|^2\right]$$

$$D_x^n f\cdot g = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

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$$f \left[iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f \right] + \left[3i\beta \left(\frac{g}{f} f_x - g_x \right) - \alpha g \right]$$

$$\times \left[D_x^2 f \cdot f + 2\kappa \left|g\right|^2 \right]$$

$$D_x^n f \cdot g = \sum\nolimits_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k \frac{\partial^{n-k}}{\partial x^{n-k}} f(x) \frac{\partial^k}{\partial x^k} g(x)$$

$$iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f = 0, \qquad D_x^2 f \cdot f = -2\kappa \left|g\right|^2$$

Nonlocality in Hirota's direct method

Bilinearisation of the local Hirota equation ($q = g/f$)

$$f^3 \left[iq_t + \alpha q_{xx} - 2\kappa\alpha \left| q \right|^2 q + i\beta \Big(q_{xxx} - 6\kappa \left| q \right|^2 q_x \Big) \right] =$$

$$f \left[iD_t g \cdot f + \alpha D_x^2 g \cdot f + i\beta D_x^3 g \cdot f \right] + \left[3i\beta \left(\frac{g}{f} f_x - g_x \right) - \alpha g \right]$$

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Solve by formal power series that becomes **exact**

$$f(x,t) = \sum_{k=0}^{\infty} \varepsilon^{2k} f_{2k}(x,t), \quad \text{and} \quad g(x,t) = \sum_{k=1}^{\infty} \varepsilon^{2k-1} g_{2k-1}(x,t)$$

Bilinearisation of the nonlocal Hirota equation

$$
\begin{aligned}
&f^3\tilde{f}^* \left[iq_t + \alpha q_{xx} + 2\alpha\tilde{q}^* q^2 - \delta(q_{xxx} + 6q\tilde{q}^* q_x) \right] = \\
&f\tilde{f}^* \left[iD_t g\cdot f + \alpha D_x^2 g\cdot f - \delta D_x^3 g\cdot f \right] + \left(\frac{3\delta}{f} D_x g\cdot f - \alpha g \right) \\
&\times \left(\tilde{f}^* D_x^2 f\cdot f - 2fg\tilde{g}^* \right)
\end{aligned}
$$

not bilinear yet

$$
iD_t g\cdot f + \alpha D_x^2 g\cdot f - \delta D_x^3 g\cdot f = 0, \quad \tilde{f}^* D_x^2 f\cdot f = 2fg\tilde{g}^*
$$

Bilinearisation of the nonlocal Hirota equation

$$f^3\tilde{f}^*\left[iq_t+\alpha q_{xx}+2\alpha\tilde{q}^*q^2-\delta(q_{xxx}+6q\tilde{q}^*q_x)\right]=$$

$$f\tilde{f}^*\left[iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f\right]+\left(\frac{3\delta}{f}D_xg\cdot f-\alpha g\right)$$

$$\times\left(\tilde{f}^*D_x^2f\cdot f-2fg\tilde{g}^*\right)$$

not bilinear yet

$$iD_tg\cdot f+\alpha D_x^2g\cdot f-\delta D_x^3g\cdot f=0,\qquad \tilde{f}^*D_x^2f\cdot f=2fg\tilde{g}^*$$

introduce additional auxiliary function

$$D_x^2f\cdot f=hg,\qquad \text{and}\qquad 2f\tilde{g}^*=h\tilde{f}^*$$

Solve again formal power series that becomes **exact**

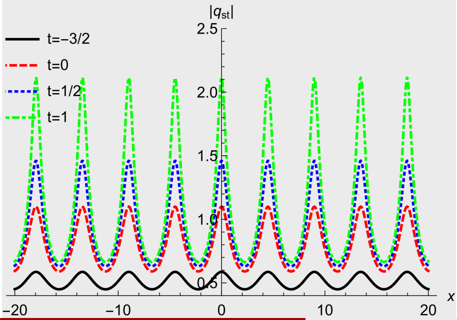
$$h(x,t)=\sum_k\varepsilon^kh_k(x,t).$$

Two-types of nonlocal solutions (one-soliton)

Truncated expansions: $f = 1 + \varepsilon^2 f_2, \quad g = \varepsilon g_1, \quad h = \varepsilon h_1$

$$
\begin{aligned}
0 &= \varepsilon[i(g_1)_t + \alpha(g_1)_{xx} - \delta(g_1)_{xxx}] \\
&\quad + \varepsilon^3[2(f_2)_x(g_1)_x - g_1[(f_2)_{xx} + i(f_2)_t] + if_2[(g_1)_t + i(g_1)_{xx}]] \\
0 &= \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2f_2(f_2)_{xx} - 2(f_2)_x^2\right] \\
0 &= \varepsilon[2\tilde{g}_1^* - h_1] + \varepsilon^3\left[2f_2\tilde{g}_1^* - \tilde{f}_2^* h_1\right]
\end{aligned}
$$

Standard solution, solve six equations independently, then $\varepsilon \to 1$



$$
q_{st}^{(1)} = \frac{\lambda(\mu - \mu^*)^2 \tau_{\mu,\gamma}}{(\mu - \mu^*)^2 + |\lambda|^2 \tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}
$$

$$
\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}
$$

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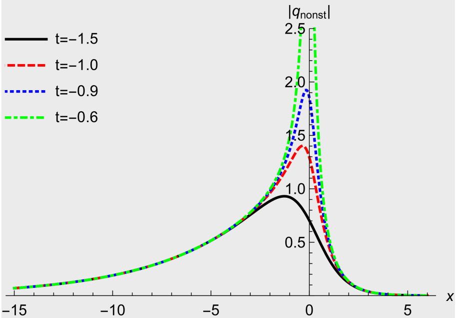
0 &= \varepsilon^2[2(f_2)_{xx} - g_1 h_1] + \varepsilon^4\left[2f_2(f_2)_{xx} - 2(f_2)_x^2\right] \\
0 &= [2\tilde{g}_1^* - h_1] + \left[2f_2\tilde{g}_1^* - \tilde{f}_2^* h_1\right]
\end{aligned}
$$

Nonstandard solution, solve five equations, last one for $\varepsilon = 1$



$$
q_{\rm nonst}^{(1)} = \frac{(\mu+\nu)\tau_{\mu,i\gamma}}{1 + \tau_{\mu,i\gamma}\tilde{\tau}^*_{-\nu,-i\theta}}.
$$

$$
\tau_{\mu,\gamma}(x,t) := e^{\mu x + \mu^2(i\alpha - \beta\mu)t + \gamma}
$$

Two-soliton solution

Truncated expansions:

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4, \quad g = \varepsilon g_1 + \varepsilon^3 g_3, \quad h = \varepsilon h_1 + \varepsilon^3 h_3$$

$$q_{\mathrm{nl}}^{(2)}(x,t) = \frac{g_1(x,t) + g_3(x,t)}{1 + f_2(x,t) + f_4(x,t)}$$

$$\begin{aligned}
g_1 &= \tau_{\mu,\gamma} + \tau_{\nu,\delta} \\
g_3 &= \frac{(\mu-\nu)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma} + \frac{(\mu-\nu)^2}{(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
f_2 &= \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}}{(\mu-\mu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\mu,\gamma}}{(\nu-\mu^*)^2} + \frac{\tau_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}}{(\mu-\nu^*)^2} + \frac{\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta}}{(\nu-\nu^*)^2} \\
f_4 &= \frac{(\mu-\nu)^2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu-\mu^*)^2(\mu-\nu^*)^2(\nu-\nu^*)^2}\tau_{\mu,\gamma}\tilde{\tau}^*_{\mu,\gamma}\tau_{\nu,\delta}\tilde{\tau}^*_{\nu,\delta} \\
h_1 &= 2\tilde{\tau}^*_{\mu,\gamma} + 2\tilde{\tau}^*_{\nu,\delta} \\
h_3 &= \frac{2(\mu^*-\nu^*)^2}{(\mu-\mu^*)^2(\nu^*-\mu)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\mu,\gamma} + \frac{2(\mu^*-\nu^*)^2}{(\mu^*-\nu)^2(\nu-\nu^*)^2}\tilde{\tau}^*_{\mu,\gamma}\tilde{\tau}^*_{\nu,\delta}\tau_{\nu,\delta}
\end{aligned}$$

Two-soliton solution

Truncated expansions:

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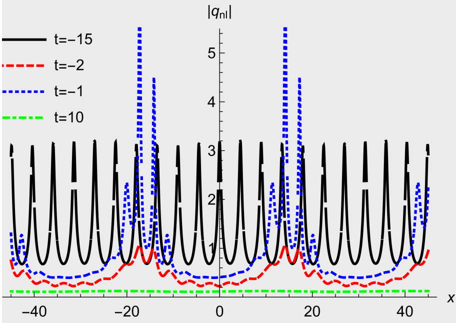
$$q_{\text{nl}}^{(2)}(x,t) = \frac{g_1(x,t) + g_3(x,t)}{1 + f_2(x,t) + f_4(x,t)}$$

Nonlocal regular two-soliton solution



Stability analysis – generalities

Consider systems of the general form

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi / 2 - V(\varphi)$$

Euler-Lagrange equation

$$\ddot{\varphi} - \varphi'' + \partial V(\varphi)/\partial\varphi = 0$$

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Linearise the Euler-Lagrange equation with $\varphi \to \varphi_s + \varepsilon \chi, \ \varepsilon \ll 1$

$$\ddot{\varphi}_s - \varphi_s'' + \left. \frac{\partial V(\varphi)}{\partial \varphi} \right|_{\varphi_s} + \varepsilon \left(\ddot{\chi} - \chi'' + \chi \left. \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \right|_{\varphi_s} \right) + \mathcal{O}(\varepsilon^2) = 0$$

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With $\chi(x,t) = e^{i\lambda t}\Phi(x) \Rightarrow$ Sturm-Liouville eigenvalue problem

$$-\Phi_{xx} + \left.\frac{\partial^2 V(\varphi)}{\partial \varphi^2}\right|_{\varphi_s} \Phi = \lambda^2 \Phi,$$

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Energies

$$E[\varphi] = \int_{-\infty}^{\infty} dx \varepsilon(\varphi), \qquad \varepsilon(\varphi) = \left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\varphi')^2 + V(\varphi) \right),$$

Perturbed energy:

$$\begin{aligned} E[\varphi_s + \chi] \ = \ & E[\varphi_s] + \int_{-\infty}^{\infty} dx \left[\left(\left. \frac{\partial V(\varphi)}{\partial \varphi} \right|_{\varphi_s} - \varphi_s'' \right) \chi \right. \\ & \left. + \frac{\chi}{2} \left(\frac{\dot{\chi}^2}{\chi} - \chi'' + \chi \left. \frac{\partial^2 V(\varphi)}{\partial \varphi^2} \right|_{\varphi_s} \right) \right] \\ & + \chi(\chi' + \varphi_s') \Big|_{-\infty}^{\infty} + \mathcal{O}(\varepsilon^3) \end{aligned}$$

F. Correa, A. Fring, T. Taira, *arXiv:2110.06825* (2021)

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The Bullough-Dodd model

$$\mathcal{L}_{\mathrm{BD}} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - e^{\varphi} - \frac{1}{2}e^{-2\varphi} + \frac{3}{2} \qquad \text{with} \quad \varphi \in \mathbb{C}$$

Euler-Lagrange equation: $\ddot{\varphi} - \varphi'' + e^{\varphi} - e^{-2\varphi} = 0$

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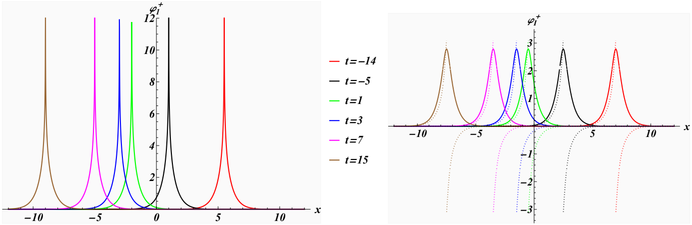
Type I sol.: $\varphi_I^{\pm}(x,t) = \ln\left[\frac{\cosh\left(\beta + \sqrt{k^2-3}t + kx\right) \pm 2}{\cosh\left(\beta + \sqrt{k^2-3}t + kx\right) \mp 1}\right], \quad \beta \in \mathbb{C}$

$\varphi_I^+, |k| > \sqrt{3}$:



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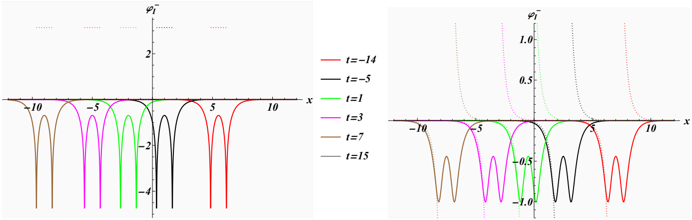
$\varphi_I^-, |k| > \sqrt{3}$:



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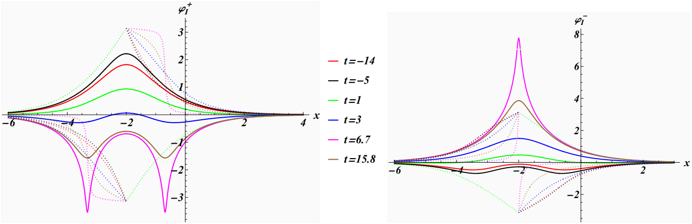
$\varphi_I^{\pm}, |k| < \sqrt{3}$:



Sturm-Liouville auxiliary problem

with potential

$$
\begin{aligned}
V_1^+(x,\beta) &= 1 - \frac{3}{1-\cosh\left(\beta+\sqrt{3}x\right)} + \frac{8\sinh^4\left[\frac{1}{2}\left(\beta+\sqrt{3}x\right)\right]}{\left[2+\cosh\left(\beta+\sqrt{3}x\right)\right]^2} \\
V_1^-(x,\beta) &= V_1^+(x,\beta-i\pi)
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$$

Darboux transformation $\Rightarrow$ exactly solvable partner potential

$$V_2 = 3 - \frac{3}{2}\operatorname{sech}^2\left(\frac{\beta}{2}+\frac{\sqrt{3}x}{2}\right).$$

We find one bound state with $\lambda = 3/2 \Rightarrow$ **the solution is stable.**

Sturm-Liouville auxiliary problem

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$$V_1^+(x,\beta) = 1 - \frac{3}{1-\cosh\left(\beta+\sqrt{3}x\right)} + \frac{8\sinh^4\left[\frac{1}{2}\left(\beta+\sqrt{3}x\right)\right]}{\left[2+\cosh\left(\beta+\sqrt{3}x\right)\right]^2}$$

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Similarly for type II solutions

Sturm-Liouville auxiliary problem

with potential

$$\begin{aligned} V_1^+(x,\beta) &= 1 - \frac{3}{1-\cosh\left(\beta+\sqrt{3}x\right)} + \frac{8\sinh^4\left[\frac{1}{2}\left(\beta+\sqrt{3}x\right)\right]}{\left[2+\cosh\left(\beta+\sqrt{3}x\right)\right]^2} \\ V_1^-(x,\beta) &= V_1^+(x,\beta-i\pi) \end{aligned}$$

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Also the nonlocal solutions are found to be stable,
see J. Cen, F. Correa, F., A. Fring, T. Taira, *Stability in integrable nonlocal nonlinear equations* Physics Letters A, 435, (2022) 128060

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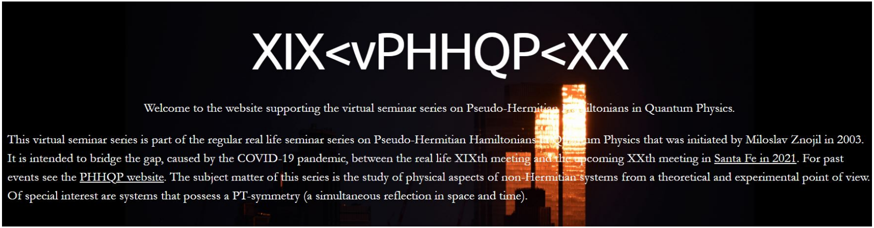

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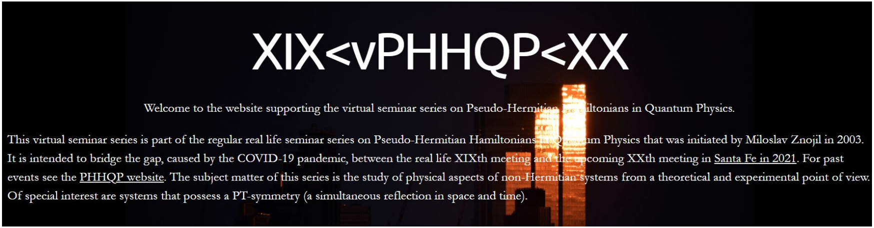
virtual seminar Pseudo-Hermitian Hamiltonians in Quantum Physics



https://vphhqp.com

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◂ Previous issue   Next issue ▸

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Thank you for your attention