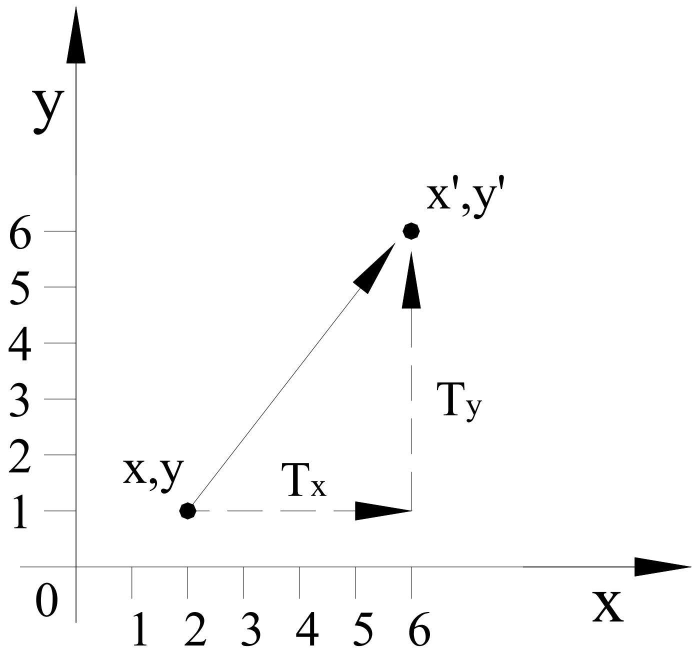
Transformations

- Matrix revision
- Translation
- Scale
- Rotation
- Reflection
- Matrix representation
- Worked example and problems

Translation



$$x' = x + T_x$$

$$y' = y + T_y$$

$$P' = P + T$$

In Matrix form

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} + \begin{bmatrix} T_x & T_y \end{bmatrix}$$

Why matrix form? Standard transformation matrices

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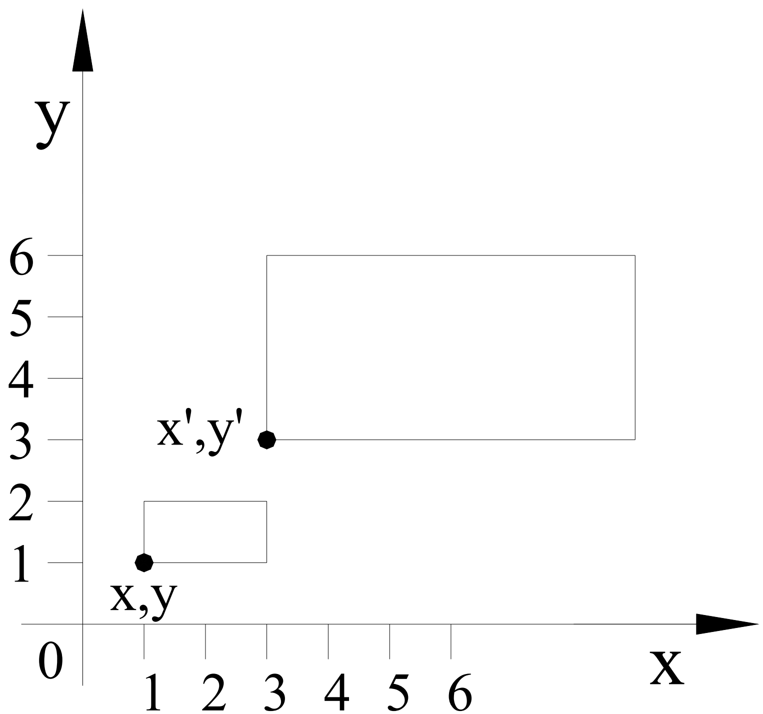
Scaling



$$x' = x \cdot S_x$$

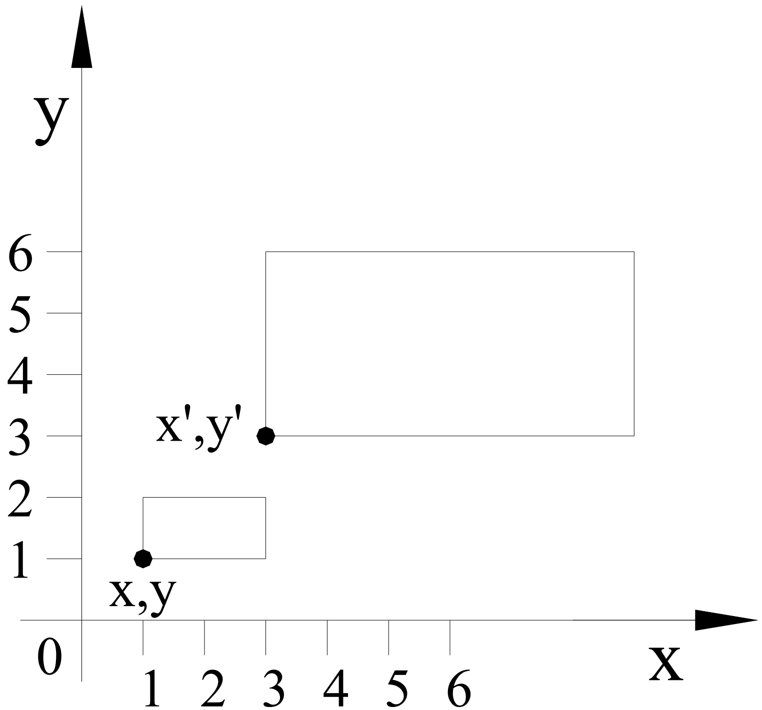
$$y' = y \cdot S_y$$

$$P' = P \cdot S$$

In Matrix form

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

Calculating Scale Factors



$$x = 1$$

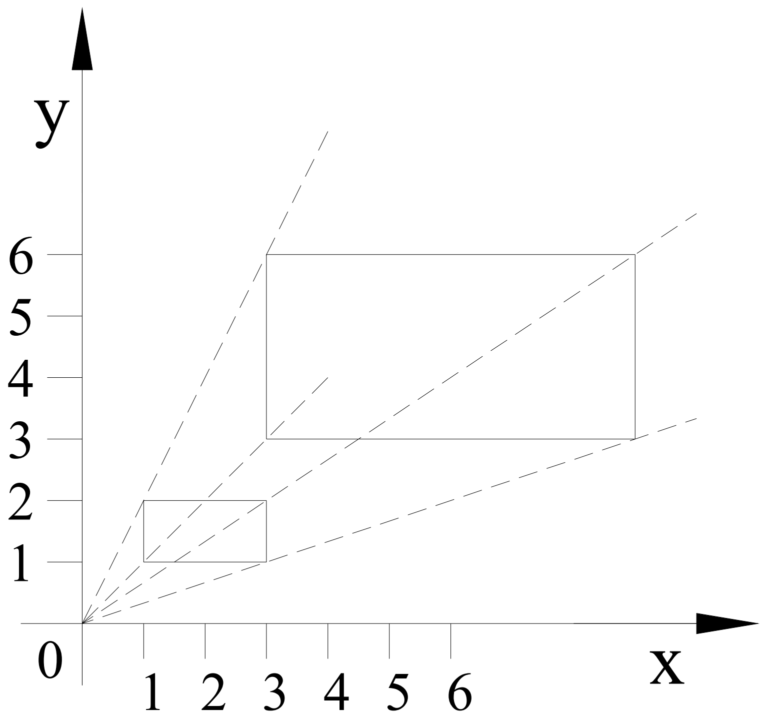
$$x' = 3$$

$$x' = x \cdot S_x$$

$$\Rightarrow S_x = \frac{x'}{x} = \frac{3}{1} = 3$$

What is the value of S_y?

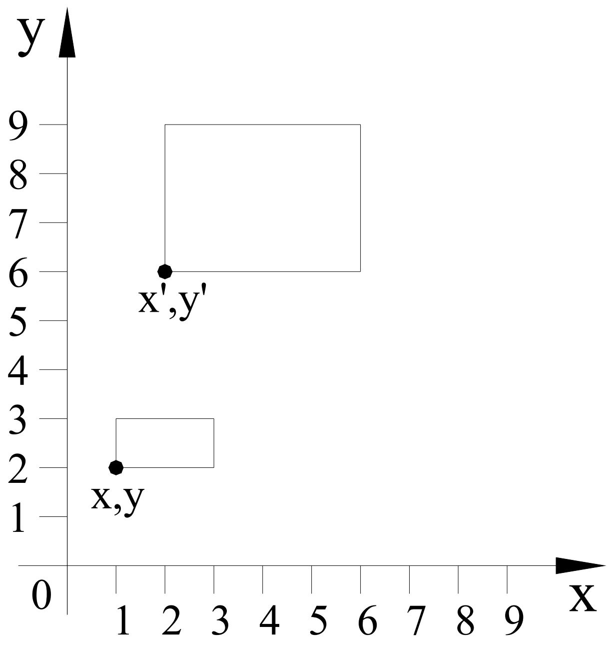
Origin for scaling



All scaling in these examples is relative to the co-ordinate origin (0,0) for simplicity.

Note: It is possible to scale from any point.

Unequal Scale Factors



$$x = 1$$

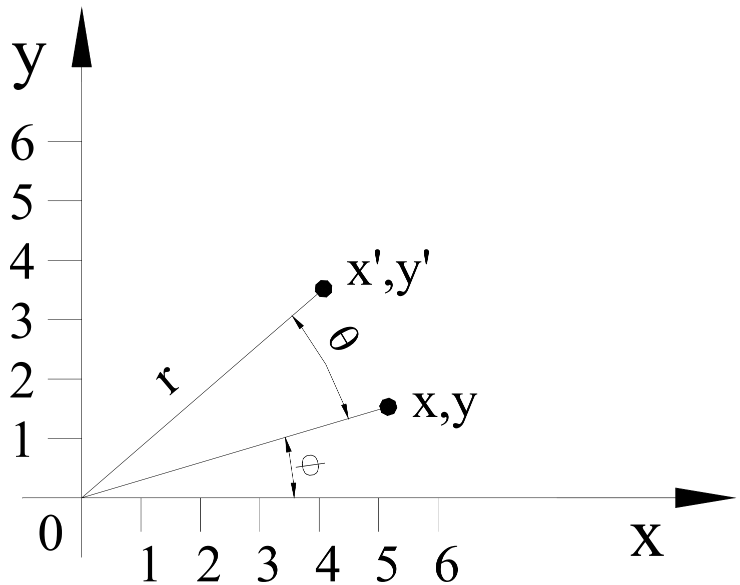
$$x' = 2$$

We know $x' = x \cdot S_x$

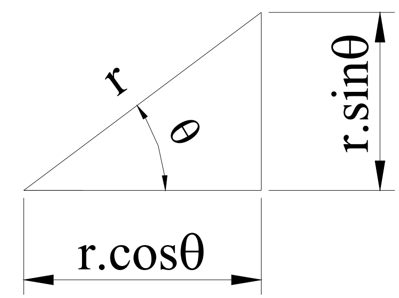
$$\Rightarrow S_x = \frac{x'}{x} = \frac{2}{1} = 2$$

What is the value of S_y?

Rotation



First remember



$$x' = r.\cos(\phi + \theta) = r.\cos\phi\cos\theta - r.\sin\phi\sin\theta$$

$$y' = r.\sin(\phi + \theta) = r.\cos\phi\sin\theta + r.\sin\phi\cos\theta$$

This uses the trigonometric identities.

Rotation 2

We know $x = r\cos\phi$ and $y = r\sin\phi$

Substituting gives

$$x' = x.\cos\theta - y\sin\theta$$

$$y' = x.\sin\theta + y.\cos\theta$$

In matrix form

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

or

$$P' = P \cdot R$$

For these calculations all rotations are relative to the co-ordinate origin (0,0) for simplicity.

Note: The rotations may be relative to any point.

The formula would have to be adjusted to take account of this.

Matrix Representations

All transformations are of the type

$$P' = M_1 \cdot P + M_2$$

Where M is a matrix

Convenient to lose M_2 term

– then all transformations achieved by matrix multiplication

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_y & 1 \end{bmatrix} = \begin{bmatrix} x + 0 + T_x & 0 + y + T_y & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} x + T_x & y + T_y & 1 \end{bmatrix}$$

Homogeneous co-ordinate (not z value)

Rotation and Scaling Matrices

Using the homogeneous co-ordinate

The scaling matrix becomes

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \cdot \begin{bmatrix} S_X & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix becomes

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

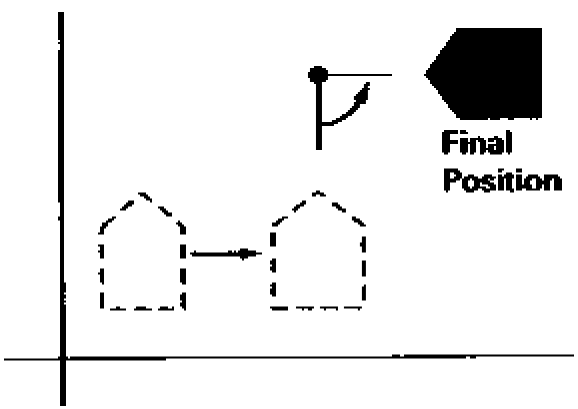
Concatenation

Matrix multiplication is asociative

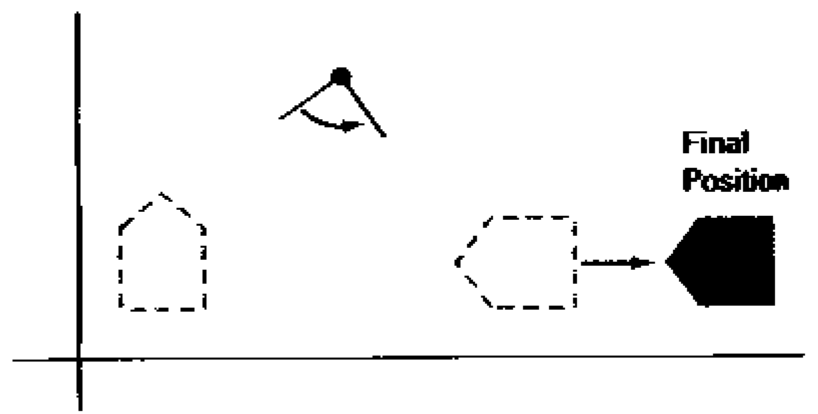
$$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$$

BUT transformations are not necessarily commutative

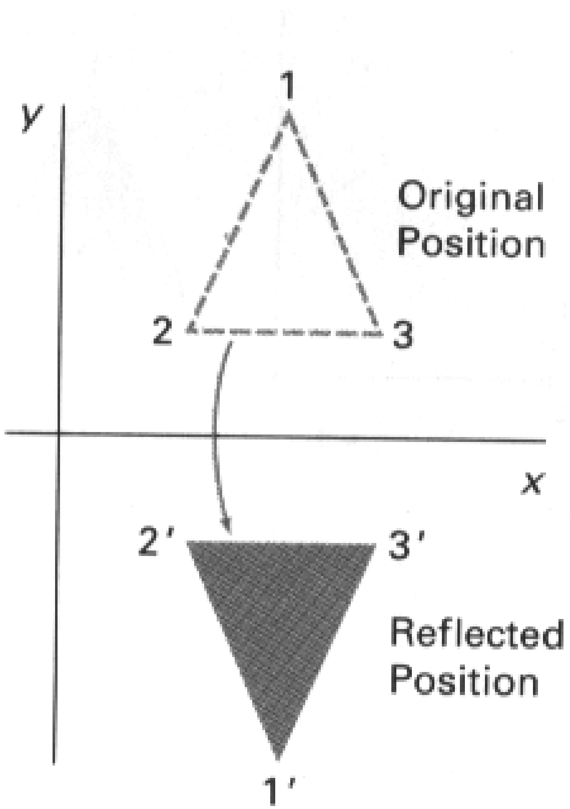
Translation then rotation



Rotation then translation



Reflection

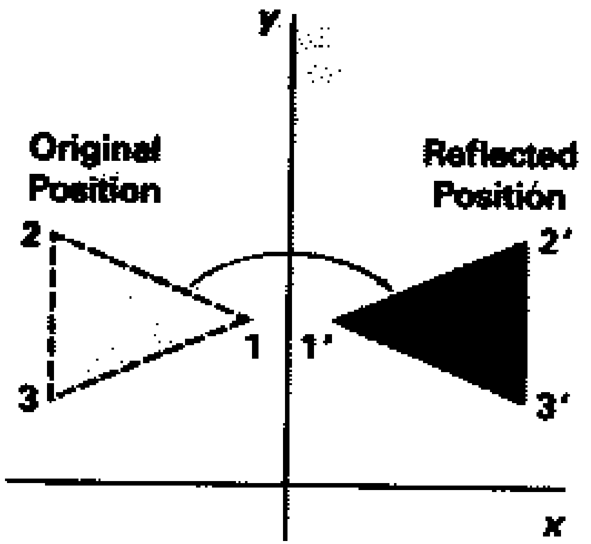


In this transformation the x values remain the same but the y values flip.

Transformation matrix for reflection about the x axis (y=0) is shown below

$$[x' \quad y' \quad 1] = [x \quad y \quad 1] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection 2



Transformation matrix for reflection about the y axis (x=0) is shown below

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Worked example

Translate the point 2,2 by 4 in the x direction and 3 in the y direction. Evaluate your answer using matrices.

First - sketch the problem

Worked example cont.

We know

$$P' = P \cdot T$$

Where T is the translation matrix

So

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ T_x & T_y & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

Worked example cont.

$$= \begin{bmatrix} 6 & 5 & 1 \end{bmatrix}$$

After the translation the point will be at 6,5.

Problems - use matrices

1. Translate the point 6,2 by 10 in the x direction and 3 in the y direction.
2. Translate the line from 1,2 to 4,5 by 7 in the x direction and 3 in the y direction.

   Hint: put both sets of co-ordinates in one matrix as shown below

$$\begin{bmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix}$$

3. Reflect the point 6,2 in the x axis.

More problems

4 Translate the triangle (2,2) (2,5) and (5,5) by 4 in the x direction and 2 in the y direction.

5 Scale the triangle in Qu4 by a scale factor of 2. Scale relative to the co-ordinate origin.

6 Scale the triangle in Qu 4 by a scale factor of 3 in the x direction and 4 in the y direction. Scale relative to the co-ordinate origin.

7 Rotate the triangle in Qu 4 by 90° anti-clockwise about 0,0.

8 Rotate the triangle in Qu 4 by 180 ° about 0,0.

9 Reflect the triangle in Qu 4 in the y axis.

Answers

1 16,5

2 (8,5) and (11,8)

3 6,-2

For the triangle problems below you may find it useful to check your answers using squared paper

4 (6,4) (6,7) and (9,7)

5 (4,4) (4,10) and (10,10)

6 (6,8) (6,20) and (15,20)

7 (-2,2) (-5,2) and (-5,5)

8 (-2,-2) (-2,-5) and (-5,-5)

9 (-2,2) (-2,5) and (-5,5)