Varieties for modules over commutative rings?

Srikanth B. Iyengar

University of Utah, Salt Lake City

Isle of Skye, 25th June 2015

Pacific Institute *for the* Mathematical Sciences

PIMS Summer School and Workshop on Geometric and Topological Aspects of the Representation Theory of Finite Groups

Summer School	27-30 July, 2016
Workshop	1-5 August, 2016

Earth Sciences Building
University of British Columbia

SUMMER SCHOOL SPEAKERS

- Eric Friedlander (U. Southern California)
- Jesper Grodal (U. Copenhagen)
- Radha Kessar (City U., London)
- Peter Symonds (U. Manchester)

TENTATIVE WORKSHOP SPEAKERS:

- Alejandro Adem (U. British Columbia)
- Luchezar L. Avramov (U. Nebraska-Lincoln)
- Paul Balmer (U. California, Los Angeles)
- David Eisenbud (U. California, Berkeley)
- Bob Guralnick (U. Southern California)
- Alexander Kleschev (U. Oregon)
- Henning Krause (U. Bielefeld)
- Markus Linckelmann (City U., London)
- Dan Nakano (U. Georgia, Athens)
- Greg Stevenson (U. Bielefeld)
- Brooke Shipley (U. Illinois, Chicago)
- Burt Totaro (U. California, Los Angeles)

ORGANIZERS: Jon F. Carlson (U. Georgia), Srikanth B. Iyengar (U. Utah) and Julia Pevtsova (U. Washington)

WEBSITE: *www.pims.math.ca/scientific-event/160727-psswgatartofg*

These events will be an occasion to celebrate Dave Benson's 60th birthday and his distinguished mathematical career.

Outline

- Modules over commutative rings
- A theorem (really, a lemma)
- An example/application
- A sketch of a proof
- Varieties for modules?

The essence of mathematics is proving theorems—and so, that is what mathematicians do: They prove theorems. But to tell the truth, what they really want to prove, once in their lifetime, is a Lemma, like the one by Fatou in analysis, the Lemma of Gauss in number theory, or the Burnside-Frobenius Lemma in combinatorics.

Now what makes a mathematical statement a true Lemma? First, it should be applicable to a wide variety of instances, even seemingly unrelated problems. Secondly, the statement should, once you have seen it, be completely obvious. The reaction of the reader might well be one of faint envy: Why haven't I noticed this before? And thirdly, on an esthetic level, the Lemma—including its proof—should be beautiful!

–Aigner and Ziegler (Proofs from The Book).

Modules over commutative rings

Throughout, k will be a field;

$x_1, \ldots, x_d$ (commuting) indeterminates over k;

$P := k[x_1, \ldots, x_d]$ the ring of polynomials in $x_1, \ldots, x_d$.

$R := P/I$ where I is a homogeneous ideal in P.

M an R-module (assume finitely generated, for simplicity).

Polynomial rings

First case: $I = 0$

So we are talking about modules over $P = k[x_1, \dots, x_d]$.

We can pretend we "understand" all P-modules.

Some justification: for any P-module M, one has

$$\operatorname{proj\,dim}_P M \leq d$$

This is Hilbert's Syzygy Theorem.

In other words, $\Omega_P^d(M)$ is projective (even free).

Up shot: high syzygies of M are not complicated.

Notation: $\operatorname{proj\,dim}_R M < \infty$ means $\Omega_R^i M$ projective for $i \gg 0$.

Hypersurface rings

$R = P/(f)$ with $f \neq 0$ a homogenous polynomial of degree ≥ 2.

M an R-module

Eisenbud: $\Omega_R^d(M)$ can be described by a matrix factorisation of f:

There exist $n \times n$ matrices A and B over P such that

$$AB = f\, I_n = BA$$

And $M := \operatorname{Coker}(P^n \xrightarrow{A} P^n)$.

Up shot: Module theory over R is reduced to linear algebra.

Can be quite complicated, but it is at least concrete.

In fact: For some purposes, we can reduce to the case

$$R = k[x]/(x^t)$$

General case

$R = P/I$ where $I \subseteq (x_1, \ldots, x_d)^2$ and M an R-module

For any $f \in I$, one has $P/(f) \twoheadrightarrow R$. Write

$$M{\downarrow_{P/(f)}} := M \text{ viewed as an } P/(f)\text{-module.}$$

Idea: Study M by studying the family $\{M{\downarrow_{P/(f)}} \mid f \in I\}$.

Reminiscent of:

cyclic shifted subgroups for elementary abelian groups
– Carlson

π-points for finite group schemes
– Friedlander, Pevtsova

generic hypersurfaces for complete intersections
– Avramov, Buchweitz, Jorgensen

Question: How do properties of $M{\downarrow_{P/(f)}}$ vary as we vary f?

Keep in mind: $M{\downarrow_{P/(f)}}$ is a module over $P/(f)$, so the rings vary.

Jorgensen (2002, Pacific Journal of Math.) proved:

Theorem

If f, g in I are such that $f - g$ is in $(x_1, \ldots, x_d)I$, then

$$\operatorname{proj\,dim}(M{\downarrow_{P/(f)}}) < \infty \iff \operatorname{proj\,dim}(M{\downarrow_{P/(g)}}) < \infty$$

Said otherwise: The property $\operatorname{proj\,dim}(M{\downarrow_{P/(f)}}) < \infty$ depends only on the residue class of f in

$$\frac{I}{(x_1, \ldots, x_d)I} \cong k^c$$

where c is the number of generators of I.

Plan: To discuss one application and a sketch of a (new) proof.

First, a simple example.

Example

$R = k[x, y, z]/(x^p, y^q, z^r)$ with $p, q, r \geq 2$

For any polynomials a, b in (x, y, z), the Theorem yields

$$\operatorname{proj\,dim}(M{\downarrow}_{P/(x^p)}) < \infty \iff \operatorname{proj\,dim}(M{\downarrow}_{P/(x^p+ay^q+bz^r)}) < \infty$$

Nothing sacrosanct about x^p, y^q, z^r; any forms would do.

Another example

Suppose now $\mathrm{char}(k) = p > 0$ and $R = k[x,y,z]/(x^p, y^p, z^p)$.

$$
\begin{tikzcd}
 & \frac{k[x,y,z]}{(x^p)} \arrow[dr] & \\
\frac{k[t]}{(t^p)} \arrow[ur, "t \mapsto x"] \arrow[rr, "t \mapsto x"] & & \frac{k[x,y,z]}{(x^p, y^p, z^p)}
\end{tikzcd}
$$

The map $k[t]/(t^p) \longrightarrow k[x,y,z]/(x^p)$, is a polynomial extension, so one gets the first implication below:

$$
\begin{aligned}
\operatorname{proj\,dim}(M{\downarrow}_{k[x,y,z]/(x^p)}) < \infty &\iff \operatorname{proj\,dim}(M{\downarrow}_{k[t]/(t^p)}) < \infty \\
&\iff M{\downarrow}_{k[t]/(t^p)} \text{ is free}
\end{aligned}
$$

The second implication is clear.

Example - continued

Recall $\mathrm{char}(k) = p > 0$ and $R = k[x,y,z]/(x^p, y^p, z^p)$.
Consider the diagram:

$$\begin{tikzcd}
 & \frac{k[x,y,z]}{(x^p)} \arrow[dr, two heads] & & \frac{k[x,y,z]}{(x^p+y^pz^p)} \arrow[dl, two heads] & \\
\frac{k[t]}{(t^p)} \arrow[ur] \arrow[rr, "t\mapsto x"] & & \frac{k[x,y,z]}{(x^p,y^p,z^p)} & & \frac{k[u]}{(u^p)} \arrow[ul] \arrow[ll, "x+yz\leftarrow u"']
\end{tikzcd}$$

The triangle on the right is where we need $\mathrm{char}(k) = p > 0$. Recall

$$\operatorname{proj\,dim}(M\downarrow_{k[x,y,z]/(x^p)}) < \infty \iff M\downarrow_{k[t]/(t^p)} \text{ is free}$$

In the same vein, one gets

$$\operatorname{proj\,dim}(M\downarrow_{k[x,y,z]/(x^p+y^pz^p)}) < \infty \iff M\downarrow_{k[u]/(u^p)} \text{ is free}$$

Theorem gives: $M\downarrow_{k[t]/(t^p)}$ is free if and only if $M\downarrow_{k[u]/(u^p)}$ is free.

Linear algebra translation

k a field of characteristic $p > 0$ and V a k-vector space.

$\alpha, \beta, \gamma \colon V \to V$ are k-linear maps that are p-nilpotent:

$$\alpha^p = 0\,, \quad \beta^p = 0\,, \quad \text{and} \quad \gamma^p = 0$$

Assume that α, β, and γ commute with each other.

The previous conclusion translates to:

Corollary

$\operatorname{rank}(\alpha)$ *is maximal if and only if* $\operatorname{rank}(\alpha + \beta\gamma)$ *is maximal.*

– Carlson, Suslin, Friedlander, Pevtsova

– This is a key ingredient in the theory of rank varieties.

The Koszul DG algebra

$P = k[x_1, \ldots, x_d]$ and $R = P/I$ with $I \subseteq (x_1, \ldots, x_d)^2$.

K :=Koszul complex over R, as a differential graded algebra. Thus

$K = R\langle u_1, \ldots, u_d \rangle$, the exterior algebra on indeterminates

$$u_1, \ldots, u_d \quad \text{with} \quad |u_i| = 1 \quad \text{with}$$
$$d(u_i) = x_i \quad \text{for } i = 1, \ldots, d.$$

Fix $f \in I$ and write $f = \sum_i a_i x_i$. Then $\sum_i a_i u_i$ is a cycle in K_1:

$$d(\sum a_i u_i) = \sum a_i x_i = f = 0$$

The following map is a morphism of DG algebras:

$$\theta_f \colon k\langle z \mid d(z) = 0 \rangle \longrightarrow K \quad \text{where } \theta_f(z) = \sum a_i u_i$$

Summary: Picking $f \in I$ is tantamount to choosing a morphism

$$\theta_f \colon k\langle z \mid d(z) = 0\rangle \longrightarrow K$$

M an R-module. Then $K \otimes_R M$ is a DG module over K.

$\theta_f(K \otimes_R M) := (K \otimes_R M)\downarrow_{k\langle z\rangle}$, a DG $k\langle z\rangle$-module.

Lemma

$\operatorname{proj\,dim}_{P/(f)} M < \infty \iff$ *the DG module* $\theta_f(K \otimes_R M)$ *is perfect.*

Conclusion: whether $\operatorname{proj\,dim}_{P/(f)} M$ is finite depends only on the structure of $K \otimes_R M$ as a DG module over $k\langle z\rangle$ via θ_f.

The point is that $k\langle z\rangle$ is rather simple, as DG algebras go.

Sketch of a proof of the theorem

Theorem

If f, g in I are such that $f - g$ is in $(x_1, \dots, x_d)I$, then

$$\operatorname{proj\,dim}(M{\downarrow}_{P/(f)}) < \infty \iff \operatorname{proj\,dim}(M{\downarrow}_{P/(g)}) < \infty$$

As above, each of f and g induces a morphism of DG algebras

$$k\langle z\rangle \underset{\theta_g}{\overset{\theta_f}{\rightrightarrows}} K$$

Key point: Under the hypothesis one has

$$[\theta_f(z)] = [\theta_g(z)] \quad \text{in} \quad H_1(K).$$

Thus, θ_f and θ_g are homotopic.

Better still:

They can even be made equal, after replacing K by a quasi-isomorphic DGA.

Consequently, there is an quasi-isomorphism of DG $k\langle z\rangle$-modules

$$\theta_f^*(K \otimes_R M) \simeq \theta_g^*(K \otimes_R M)$$

Thus $\theta_f^*(K \otimes_R M)$ and $\theta_g^*(K \otimes_R M)$ are perfect simultaneously. $\square$

The proof gives more:

$M{\downarrow_{P/(f)}}$ and $M{\downarrow_{P/(g)}}$ are essentially the "same".

Varieties

$R = P/I$ where $I = (f_1, \dots, f_c)$

M a finitely generated R-module. Set

$$\mathcal{V}_R(M) := \{[a_1, \dots, a_c] \in \mathbb{P}_k^{c-1} \mid \operatorname{proj\,dim}(M\!\downarrow_{P/(a_1 f_1 + \cdots a_c f_c)}) = \infty\}$$

This is a Zariski-closed subset of $\mathbb{P}_k^{c-1}$.

When R is complete intersection, this is the cohomological variety introduced by Avramov and Buchweitz.

For general R, these are not well-understood.

For example, $\mathcal{V}_R(R)$ need not be $\varnothing$. In fact

Proposition

$\mathcal{V}_R(R) = \varnothing \iff R$ *is complete intersection.*

Next progress report: Benson’s 70th birthday!