1. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{1}{n+r}=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\frac{1}{1+\frac{r}{n}}=\int_0^1\frac{\mathrm{d}x}{1+x}=\ln(2).$

2. At first, we note that $0<e-\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\frac{1}{k!}<\frac{2}{(n+1)!}$.

Let $k_n=n!\sum\limits_{k=0}^{n}\frac{1}{k!}\underset{3}{\equiv}1+n+n(n-1)=n^2+1=3m_n+l_n$, where $l_n\neq 0$. Putting into the formula: $\lim\limits_{n\to\infty}\cos\frac{2\pi en!}{3}=\lim\limits_{n\to\infty}\cos\left(\frac{2\pi n!}{3}\left(e-\sum\limits_{k=0}^{n}\frac{1}{k!}\right)+\frac{2\pi}{3}(3m_n+l_n)\right)=\lim\limits_{n\to\infty}\cos\frac{2\pi}{3}l_n=-\frac{1}{2}.$

3. $\ln a_n=\frac{1}{2}\ln a_{n-1}+\frac{1}{2}\ln a_{n-2}$. The characteristic equation is $r^2-\frac{1}{2}r-\frac{1}{2}=0$. Thus, $r_1=1$ and $r_2=-\frac{1}{2}$ and $\ln a_n=x+y(-\frac{1}{2})^n$. From $\ln a=x-\frac{y}{2}$ and $\ln b=x+\frac{y}{4}$ then $x=\frac{\ln a+2\ln b}{3}$. Finally, $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}e^x=\sqrt[3]{ab^2}$.

4. $a^{2^n}+1$ grows asymptotically faster than 2^n, thus the limit is 0.

5. By Stolz-Cezaro, $\lim\limits_{n\to\infty}\frac{1^p+\ldots+n^p}{n^{p+1}}=\lim\limits_{n\to\infty}\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}=\frac{1}{p+1}$.

6. Let $t_n=x_n-y_n$. Since, $x_n^3+y_n^3=t_n^3+3t_n^2y_n+3t_ny_n^2+2y_n^3$, we get that $\lim\limits_{n\to\infty}3t_n^2y_n+3t_ny_n^2+2y_n^3=0$.

$3t_n^2y_n+3t_ny_n^2+2y_n^3=y_n(3(t_n+\frac{y_n}{2})^2+\frac{5}{4}y_n^2)\Leftrightarrow\frac{5}{4}y_n^4=3t_n^2y_n^2+3t_ny_n^3+2y_n^4-3y_n^2(t_n+\frac{y_n}{2})^2$. Thus, $0\le\frac{5}{4}y_n^4\le 3t_n^2y_n^2+3t_ny_n^3+2y_n^4=y_n(3t_n^2y_n+3t_ny_n^2+2y_n^3)\overset{n\to\infty}{=}0$.

Since $x_n=t_n+y_n$ we get $\lim\limits_{n\to\infty}x_n=0$

7. Let $a_n=\lambda^{2^{-(n-1)}}+\lambda^{-2^{-(n-1)}}$, then $a_{n+1}=\sqrt{a_n+2}$ and $\lim\limits_{n\to\infty}a_n-2=0$. For $a_1>2$, we have $(3-a_n)^{4^n}=((1-(a_n-2))^{\frac{1}{a_n-2}})^{4^n(a_n-2)}$. Thus, $\lim\limits_{n\to\infty}(3-a_n)^{4^n}=\lim\limits_{n\to\infty}\exp(-4^n(a_n-2))$.

$\lim\limits_{n\to\infty}4^n(\lambda^{2^{-(n-1)}}+\lambda^{-2^{-(n-1)}}-2)=\lim\limits_{n\to\infty}4^n(1+\log(\lambda)2^{-(n-1)}+\frac{1}{2}\log^2(\lambda)2^{-2(n-1)})+O(2^{-3n})+1-\log(\lambda)2^{-(n-1)}+\frac{1}{2}\log^2(\lambda)2^{-2(n-1)}+O(2^{-3n})-2)=4\log^2(\lambda)$.

By that $\lim\limits_{n\to\infty}(3-a_n)^{4^n}=\exp(-4\log^2(\lambda))$.

$a_1=\lambda+\lambda^{-1}$, thus $\lambda=\frac{a_1+\sqrt{a_1^2-4}}{2}$.

8. At first, note that $x_{n+1}<x_n$ and thus $\lim\limits_{n\to\infty}x_n=0$.

$l=\lim\limits_{n\to\infty}nx_n^2=\lim\limits_{n\to\infty}\frac{n}{\frac{1}{x_n^2}}$. By Stolz-Cesaro, $l=\lim\limits_{n\to\infty}\frac{1}{\frac{1}{x_{n-1}^2}-\frac{1}{x_n^2}}=\lim\limits_{n\to\infty}\frac{\sin^2(x_n)}{1-\frac{\sin^2(x_n)}{x_n^2}}=\lim\limits_{x\to0}\frac{\sin^2 x}{1-\frac{\sin^2 x}{x^2}}$. By replacing with Taylor series we get $l=\lim\limits_{x\to0}\frac{x^2-\frac{x^4}{3}+\frac{2x^6}{45}+\ldots}{\frac{x^2}{3}-\frac{2x^4}{45}+\ldots}=3$. Thus, $\lim\limits_{n\to\infty}\sqrt{n}x_n=\sqrt{l}=\sqrt{3}$.

9. Define $t_1=\arccos(\frac{4}{5})$, $t_{n+1}=t_n+\sin t_n$ then by induction one shows $x_n=\cos t_n$ and $y_n=\sin t_n$. Consider $f(x)=x+\sin x$. f is increasing, $f(0)=0$, $f(\pi)=\pi$ and on $[0,\pi)$ we have $f(x)>x$. Hence, $0<t_n<t_{n+1}<\pi$ for all n and $t_n\to t\in(0,\pi]$. Also, t is a fixed point, thus, $t=\pi$, $\lim\limits_{n\to\infty}t_n=\pi$, $\lim\limits_{n\to\infty}x_n=-1$ and $\lim\limits_{n\to\infty}y_n=0$.

10. Let $S_n=\sum\limits_{i=1}^{n}a_i^2$. Proving by negation we get $\lim\limits_{n\to\infty}S_n=+\infty$. Thus, $\lim\limits_{n\to\infty}\frac{S_{n+1}}{S_n}=1$.

By Stolz-Cesaro, $\lim\limits_{n\to\infty}\frac{S_n^3}{n}=\lim\limits_{n\to\infty}\frac{S_{n+1}^3-S_n^3}{1}=\lim\limits_{n\to\infty}(S_{n+1}-S_n)(S_{n+1}^2+S_{n+1}S_n+S_n^2)=\lim\limits_{n\to\infty}3a_{n+1}^2S_{n+1}^2=3$.

$\lim\limits_{n\to\infty}(3n)^{\frac{1}{3}}a_n=\lim\limits_{n\to\infty}(3na_n^3)^{\frac{1}{3}}=\lim\limits_{n\to\infty}(S_n^3a_n^3)^{\frac{1}{3}}=1.$