1. a) The inequality is equivalent to $\int\limits_a^b\int\limits_a^b (f(x)-f(y))(g(x)-g(y))\,\mathrm{d}x\,\mathrm{d}y \geq 0$.

b) Straight application of Chebyshev's inequality: $g(x) = x^n$.

c) $2\int\limits_0^1 xf^2(x)\,\mathrm{d}x\int\limits_0^1 f(y)\,\mathrm{d}y \leq \int\limits_0^1 f^2(x)\,\mathrm{d}x\int\limits_0^1 (f^2(y)+1)\,\mathrm{d}y \Rightarrow \int\limits_0^1 f^2(x)(f^2(y)-2xf(y)+1)\,\mathrm{d}x\,\mathrm{d}y \geq \int\limits_0^1 f^2(x)(f^2(y)-1)^2\,\mathrm{d}x\,\mathrm{d}y \geq 0$.

2. Suppose that $M = \max_{[0,1]}|f'|$, $m = \min_{[0,1]}|f'|$, $|f'(\alpha)| = m$ and $|f'(\beta)| = M$.

1) There exists z such that $f'(z) = 0$. Then $4\int\limits_0^1 |f(x)|\,\mathrm{d}x + \int\limits_0^1 |f''(x))|\,\mathrm{d}x \geq \int\limits_{\min(\alpha,z)}^{\max(\alpha,z)} |f''(x)|\,\mathrm{d}x = |\int\limits_\alpha^z f''(x)\,\mathrm{d}x| = M \geq f'(y)$.

2) $f(z) \geq 0$ for any z. Then $4\int\limits_0^1 |f(x)|\,\mathrm{d}x + \int\limits_0^1 |f''(x)|\,\mathrm{d}x \geq 2\int\limits_0^1 mx\,\mathrm{d}x + |\int\limits_\alpha^\beta f''(x)\,\mathrm{d}x| \geq m + M - m = M \geq f'(y)$.

3) There exists z such that $f(z) = 0$. Then $4\int\limits_0^1 |f(x)|\,\mathrm{d}x + \int\limits_0^1 |f''(x)|\,\mathrm{d}x \geq$
$4\int\limits_0^z m(z-x)\,\mathrm{d}x + 4\int\limits_z^1 m(x-z)\,\mathrm{d}x + |\int\limits_\alpha^\beta f''(x)\,\mathrm{d}x| \geq 2mz^2 + 2m(1-z)^2 + M - m \geq m + M - m = M \geq f'(y)$.

3. At first, we show that for a convex function f and $a < b < c$: $f(a-b+c) + f(b) \leq f(a) + f(c)$.
Write $b = \lambda a + (1-\lambda)c$, where $\lambda = \frac{c-a}{c-b}$. Since f is convex then:

$$f(b) = f(\lambda a + (1-\lambda)c) \leq \lambda f(a) + (1-\lambda)f(c).$$

Now, $a - b + c = (1-\lambda)a + \lambda c$ and

$$f(a-b+c) = f((1-\lambda)a + \lambda c) \leq (1-\lambda)f(a) + \lambda f(c).$$

Summing up: $f(a-b+c) + f(b) \leq f(a) + f(c)$.
The inequality is equivalent to:

$$\int\limits_1^3 f(x) + f(x+10)\,\mathrm{d}x \geq \int\limits_1^3 f(x+4) + f(x+6)\,\mathrm{d}x.$$

We take: $a = x$, $b = x+4$, $c = x+10$ and $a - b + c = x + 6$.

4. **Solution 1.** Apply AM-GM, $2\int\limits_0^1 f^4(x)\,\mathrm{d}x + \left(\int\limits_0^1 f(x)\,\mathrm{d}x\right)^4 \geq 3\sqrt[3]{\left(\int\limits_0^1 f^4(x)\,\mathrm{d}x\right)^2\left(\int\limits_0^1 f(x)\,\mathrm{d}x\right)^4}$.
By Cauchy-Schwartz Inequality:

$$\int\limits_0^1 f^3(x)\,\mathrm{d}x\int\limits_0^1 f(x)\,\mathrm{d}x \geq \left(\int\limits_0^1 f^2(x)\,\mathrm{d}x\right)^2$$

$$\int\limits_0^1 f^4(x)\,\mathrm{d}x\int\limits_0^1 f^2(x)\,\mathrm{d}x \geq \left(\int\limits_0^1 f^3(x)\,\mathrm{d}x\right)^2$$

Four times the first and two times the second give what we need.
Solution 2. AM-GM + Hölder Inequality
$\frac{2}{3}\int\limits_0^1 f^4(x)\,\mathrm{d}x + \frac{1}{3}\left(\int\limits_0^1 f(x)\,\mathrm{d}x\right)^4 \geq \left(\int\limits_0^1 f^4(x)\,\mathrm{d}x\right)^{\frac{2}{3}}\left(\int\limits_0^1 f(x)\,\mathrm{d}x\right)^{\frac{4}{3}} \geq \left(\int\limits_0^1 f^2(x)\,\mathrm{d}x\right)^2$.

5. $g(t) = 3\left(\int\limits_a^t f^2(x)\,\mathrm{d}x\right)^3 - \int\limits_a^t f^8(x)\,\mathrm{d}x$.

$$g'(t) = 9f^2(t)\left(\int_a^t f^2(x)\,\mathrm{d}x\right)^2 - f^8(t) = f^2(t)\left(3\left(\int_a^t f^2(x)\,\mathrm{d}x\right)^2 - f^6(t)\right)$$
$$\geq f^2(t)\left(3\left(\int_a^t f^2(x)f'(x)\,\mathrm{d}x\right)^2 - f^6(t)\right) = 0.$$

Thus, $g'(t) \geq 0$ and $g(b) \geq g(a) = 0$.
6. Fix $x \geq 1$. $f(y) = \frac{1}{y^2+x^2}$ is decreasing on $[0, \infty)$. Hence

$$\int_1^\infty \frac{\mathrm{d}y}{y^2+x^2} \leq \sum_{n=1}^\infty \frac{1}{n^2+x^2} \leq \int_0^\infty \frac{\mathrm{d}y}{y^2+x^2}.$$

Calulating the integrals: $\frac{\pi}{4x} \leq \frac{2}{x}(\frac{\pi}{2} - \arctan\frac{1}{x}) \leq \sum\limits_{n=1}^\infty \frac{1}{n^2+x^2} \leq \frac{\pi}{2x}$.
7. $f'(x) + f(x) > 1 \Rightarrow f'(x) + f(x) - 1 > 0 \Rightarrow e^x f'(x) + e^x f(x) - e^x > 0 \Rightarrow (e^x f(x) - e^x)' > 0$.
Thus, $g(x) = e^x(f(x) - 1)$ is increasing.
$g(1) \leq g(x) \leq g(2) \Rightarrow 0 \leq e^x(f(x)-1) \leq e^2 \Rightarrow 0 \leq f(x) - 1 \leq e^{2-x} \Rightarrow 1 \leq f(x) \leq e^{2-x} + 1 \Rightarrow$
$1 = \int\limits_1^2 \mathrm{d}x \leq \int\limits_1^2 f(x)\,\mathrm{d}x \leq \int\limits_1^2 (e^{2-x}+1)\,\mathrm{d}x = e$.
8. Note that $\int\limits_0^1 x(1-x)f'(x)\,\mathrm{d}x = x(1-x)f(x)|_0^1 - \int\limits_0^1 (x-x^2)'f(x)\,\mathrm{d}x = 0 - \int\limits_0^1 f(x)\,\mathrm{d}x + 2\int\limits_0^1 xf(x)\,\mathrm{d}x = 1$.
By Hölder Inequality,

$$1 = \int x(1-x)f'(x)\,\mathrm{d}x \leq \left((x(1-x))^{\frac{3}{2}}\,\mathrm{d}x\right)^{\frac{2}{3}}\left(\int_0^1 |f'(x)|^3\,\mathrm{d}x\right)^{\frac{1}{3}}.$$

Finally,

$$\int_0^1 |f'(x)|^3\,\mathrm{d}x \geq \left(\int_0^1 (x(1-x))^{\frac{3}{2}}\,\mathrm{d}x\right)^{-2} = B^{-2}(\frac{5}{2}, \frac{5}{2}) = \left(\frac{\Gamma(5)}{\Gamma^2(\frac{5}{2})}\right)^2 = \left(\frac{128}{3\pi}\right)^2.$$

9. Let $f(c) = \frac{1}{4}$ and, since it is maximum, $f'(c) = 0$. Then, by Taylor expansion:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(\xi)}{2}(x-c)^2 = \frac{1}{4} + \frac{f''(\xi)}{2}(x-c)^2$$

Thus, $f(0) = \frac{1}{4} + \frac{f''(\xi_1)}{2}c^2$ and $f(1) = \frac{1}{4} + \frac{f''(\xi_2)}{2}(1-c^2)$. $|f(0)| + |f(1)| = \left|\frac{1}{4} + \frac{f''(xi_1)}{2}c^2\right| + \left|\frac{1}{4} + \frac{f''(\xi_2)}{2}(1-c)^2\right| \leq \frac{1}{2} + \frac{c^2}{2}|f''(\xi_1)| + \frac{(1-c)^2}{2}|f''(\xi_2)| \leq \frac{1}{2} + \frac{1}{2}(c^2 + (1-c)^2) \leq 1$.
10. By The Mean Value Theorem,

$$|f(c)|\int_0^1 \left|x - \frac{1}{2}\right|\,\mathrm{d}x = \int_0^1 |f(x)| \cdot \left|x - \frac{1}{2}\right|\,\mathrm{d}x \geq \left|\int_0^1 f(x) \cdot \left(1 - \frac{1}{2}\right)\,\mathrm{d}x\right|$$
$$= \left|1 - \frac{1}{2} \cdot 0\right| = 1.$$

Hence, $f(c) \geq 4$.

If $f(c) = 4$ is maximum, then in the first equality $|f(x)|$ should be equal to 4 for all x, which is impossible.

11. We get several equalities.

$$\int_a^b x^2 f''(x)\,\mathrm{d}x = \left|x^2 f'(x)\right|_a^b - 2\int_a^b x f'(x)\,\mathrm{d}x$$
$$= b^2 f'(b) - a^2 f'(a) - 2\int_a^b x f'(x)\,\mathrm{d}x.$$

$$\int_a^b x f''(x)\,\mathrm{d}x = \left|x f'(x)\right|_a^b - \int_a^b f'(x)\,\mathrm{d}x = b f'(b) - a f'(a) + 0.$$

$$\int_a^b f''(x)\,\mathrm{d}x = f'(b) - f'(a).$$

Then we sum them up with the proper coefficients.

$$\int_a^b (a+b)x f''(x) - ab f''(x) - x^2 f''(x)\,\mathrm{d}x = (a+b)(bf'(b) - af'(a)) - ab(f'(b) - f'(a))$$
$$- (b^2 f'(b) - a^2 f'(a)) + 2\int_a^b x f'(x)\,\mathrm{d}x$$
$$= 2\int_a^b x f'(x)\,\mathrm{d}x$$

Thus, let $g(x) = \frac{(a+b)x - ab - x^2}{2}$, then by Cauchy-Schwartz Inequality:

$$\left(\int_a^b x f'(x)\,\mathrm{d}x\right)^2 = \left(\int_a^b g(x) f''(x)\,\mathrm{d}x\right)^2 \le \int_a^b g^2(x)\,\mathrm{d}x \int_a^b (f''(x))^2\,\mathrm{d}x = \frac{(b-a)^5}{120}\int_a^b (f''(x))^2\,\mathrm{d}x.$$